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Torus structure on graphs and twisted partition functions for minimal and affine models

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Abstract

Using the Ocneanu quantum geometry of ADE diagrams (and of other diagrams belonging to higher Coxeter–Dynkin systems), we discuss the classification of twisted partition functions for affine and minimal models in conformal field theory and study several examples associated with the WZW, Virasoro and $\mathcal{W}_3$ cases.

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1. Introduction

1.1. Purpose and structure of this article

One of the purposes of our article is to present a discussion and a classification of twisted partition functions for conformal field theories associated with minimal models and affine models of type ADE, as well as some of their generalizations associated with diagrams belonging to higher Coxeter–Dynkin systems. The whole discussion is based on the quantum geometry of these diagrams. Since the graphs themselves provide the necessary

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combinatorial data, we shall avoid as much as possible to make any explicit use of the theory of affine Lie algebras (or of their finite dimensional counterparts). Actually, we shall not use much information coming from *conformal field theory*, so that our presentation should be understood by readers with different backgrounds.

Many mathematical tools used in the study of the quantum geometry of graphs were introduced by Ocneanu (in the context of operator algebras) and later "explained" or adapted in various contexts (for instance CFT but not only) by several authors; this information is scattered in publications of very different nature. Our presentation starts from very elementary concepts and shows how one can calculate many (quantum) geometrical quantities of interest by using rather straightforward algorithms. From the data encoded in ADE diagram or their generalizations, we remind the reader how the corresponding quantum geometry is related to the (twisted or not) partition functions in affine models. We then move to minimal models in particular the unitary ones, discuss the relation with graphs and give various examples (Ising, Potts and the exceptional E_6–A_{10} model). We also consider twisted $\mathcal{W}_3$-minimal models.

Our discussion of twisted partition functions for minimal models can be summarized as follows: to a pair $(G^{(1)}, G^{(2)})$ of ADE Dynkin diagrams one can associate six types of sesquilinear forms on the space of Virasoro characters. These forms can be interpreted, in terms of minimal models, as partition functions in boundary conformal field theory [8] with defects. This classification rests on the possibility of introducing several "torus structures" for the two diagrams $G^{(1)}$ and $G^{(2)}$. Torus structures are parameterized by elements of a particular base in the Ocneanu algebra of quantum symmetries; a torus structure may have a single twist, two twists, or no twist at all. The interpretation of what we call torus structures in terms of defects (or twists) in a conformal field theory with boundary was proposed by Petkova and Zuber [37]. An application of these ideas to the discussion of the different types of partition functions for minimal models was presented in the publication[1] [33]. In general, twisted partition functions are not modular invariant, and we discuss what is left of this invariance in various cases. We also describe what happens when the ADE Dynkin diagrams are replaced by members of an higher Coxeter–Dynkin system (Di Francesco–Zuber diagrams in the case of SU(3)).

We want this article to be almost "self-contained" and we shall have therefore to remind the reader several facts or constructions that, in principle, can be found in the literature. For this reason we make here a short list of several specific results of the present paper, results that, to our knowledge, cannot be found elsewhere: the use of induction/restriction matrices to obtain all twisted partition functions (with one or two twists), the use of the multiplication table of the algebra of quantum symmetries $\mathrm{Oc}(G)$ in order to obtain identities between toric matrices, the 12×12 multiplication table of $\mathrm{Oc}(E_6)$, the list of toric matrices with two twists (and the corresponding partition functions) for the affine E_6 model, the behavior of these functions with respect to the action of the modular group, a general discussion of

[1] While finishing the redaction of our paper, we received the recent preprint [34]; the authors use a concept of twisted minimal model which is very similar to ours, they do not discuss the same examples (besides the Potts model) and do not consider generalized Coxeter–Dynkin systems, but they provide a nice lattice realization of the twisted SU(2) models. The two papers share therefore several features but focalize nevertheless on distinct aspects of the same general theory.

the various types of twisted partition functions for minimal models (see, however, the previous footnote), several explicit examples of twisted versions of Virasoro minimal models, for instance (A_{10}–E_6) and several examples of twisted $\mathcal{W}_3$-minimal models, for instance ($\mathcal{A}_4$–$\mathcal{E}_5$).

Although many of the results and formulae that we mention belong to the lore of CFT (in particular, affine WZW models or minimal models), we decide to adopt a presentation that uses graphs (or pairs of graphs) as primary data, so that we can avoid, as much as possible, to make use of results coming from the theory of Virasoro algebra or of affine Lie algebras; we therefore hope that the reader will find our presentation to be of independent interest.

1.2. Torus structures of Dynkin diagrams and their generalizations

Here is a brief presentation of the various structures that will be discussed later in this paper.

To a given Dynkin diagram G (or to a member of a higher Coxeter–Dynkin system) one associates the complex vector space (also called G) spanned by the vertices of this diagram. In some cases (in particular for all diagrams belonging to the A series), this vector space G possesses an associative (and commutative) multiplication law with positive *integral* structure constants and it is called the "graph algebra"; one also says that the diagram (or the corresponding vector space) admits "self-fusion". In the case of ADE diagrams, whether or not the vector space of the diagram G (with Coxeter number κ) admits self-fusion, it is anyway a module over the graph algebra of the diagram $A_{\kappa-1}$, with the same Coxeter number. More generally, i.e., for higher Coxeter–Dynkin systems, the vector space G is a module over a particular graph algebra that we call $\mathcal{A}(G)$.

Following Ocneanu [28], to every diagram G (with or without self-fusion) belonging to a Coxeter–Dynkin system, one can associate a bi-algebra[2] $\mathcal{B}G$. By using a particular scalar product, it is easier to think that $\mathcal{B}G$ is actually a bi-algebra (a vector space with two compatible associative algebra structures). There are two—usually distinct—block decompositions for this bi-algebra (see later). Blocks of the first type are labeled by points of a graph that we call $\mathcal{A}(G)$. Blocks of the second type are labeled by points of a graph that we call $\mathrm{Oc}(G)$. The vector spaces spanned by the vertices of these two graphs are themselves endowed with natural associative algebra structures that we denote by the same symbol as the graphs themselves. The algebra $\mathcal{A}(G)$, coincides, for G of type ADE, with the graph algebra of a particular member of the A family, and it is a commutative algebra, but $\mathrm{Oc}(G)$, also called "algebra of quantum symmetries" of G is not always commutative.

The algebra of quantum symmetries $\mathrm{Oc}(G)$, like the vector space G itself, comes with a particular basis and its multiplicative structure is encoded by a graph called $\mathrm{Oc}(G)$ whose vertices are in one-to-one correspondence with the distinguished generators. In the particular case where G is a member of the A series, the algebras $\mathcal{A}(G)$, $\mathrm{Oc}(G)$ and G coincide.

We call $i, j, \ldots$ the vertices of $\mathcal{A}(G)$, $a, b, \ldots$ the vertices of G and $x, y, \ldots$ the vertices of $\mathrm{Oc}(G)$. Remember that "vertices" should be thought of as elements of the various (distinguished) basis for the corresponding vector spaces. We denote by $\underline{0}$ the identity of

[2] This bi-algebra should be, technically, a weak Hopf algebra (or quantum groupoid), but this structure, as far as we know, has only been checked in a few cases, and we are not aware of any general proof (see, however, [13]).

$\mathrm{Oc}(G)$. The vector space G is a module over $\mathcal{A}(G)$, and the algebra $\mathrm{Oc}(G)$ is a *bi-module* over $\mathcal{A}(G)$; this bi-module structure is encoded by a set of matrices (toric matrices) defined as follows:

$$i \cdot x \cdot j = \Sigma_y (W_{x,y})_{ij} y.$$

A torus structure for the diagram G is (by definition) specified by the choice of a matrix $W_{x,y}$. If the dimension of $\mathrm{Oc}(G)$ is s, the number of independent toric structures is a priori s^2, but very often we may have degeneracies, in the sense that we may obtain the same toric matrix for different choices of the pair (x, y).

It is convenient to introduce the following terminology: the undeformed torus structure corresponds to the choice of the matrix $W_{\underline{0},\underline{0}}$, a deformed torus structure along one "defect line" specified by x corresponds to the choice of the matrix $W_{\underline{0},x}$ (or $W_{x,\underline{0}}$) and a deformed torus structure along two defect lines specified by x and y corresponds to the choice of the matrix $W_{x,y}$. It is convenient to set $W_x \doteq W_{x,\underline{0}}$ and in particular $W_0 \doteq W_{\underline{0},\underline{0}}$.

1.3. Frustrated (or twisted) partition functions for affine models

1.3.1. Twisted partition functions for affine models

For affine models characterized by the affine Kac–Moody algebra of type $\widehat{\mathrm{su}}(2)$ (chiral algebra), the classification of modular invariant partition functions is well known [6,7], and was shown to be in one-to-one correspondence with ADE Dynkin diagrams. More recently [28], it was shown that if the theory is associated with the Dynkin diagram G, its modular invariant partition function is given by $Z_0 = \bar{\chi} W_{\underline{0},\underline{0}} \chi$, where χ is a vector of the complex vector space[3] $\mathbb{C}^n$ and $W_{\underline{0},\underline{0}}$ is the toric matrix associated with the origin of the Ocneanu graph of the diagram G. This characterization of partition functions uses only the (quantum) geometry of the diagram G and does not refer to the theory of affine algebras; in this approach, for instance, the fact that χ could be interpreted as a character of an affine Lie algebra is not used; in particular, modular invariance is implemented by finite dimensional matrices representing $\mathrm{SL}(2, \mathbb{Z})$.

As shown in [36,37], the other partition functions of type $Z_x = \bar{\chi} W_{\underline{0},x} \chi$, or more generally $Z_{x,y} = \bar{\chi} W_{x,y} \chi$, can be interpreted as twisted partition functions in a boundary conformal field theory (boundary "of type" G), in the presence of defect lines of type x and y. A simple algorithm for the calculation of the matrices $W_{\underline{0},x}$ was presented in [9] (where the example of E_6 was chosen) and explicit results for all ADE cases are given in [11] (see also [12] for generalizations to higher Coxeter–Dynkin systems). The definition of matrices $W_{x,y}$ in [37] looks different from ours (we use the description of the bimodule structure of $\mathrm{Oc}(G)$ over $A_{\kappa-1}$) but it can be shown to be equivalent (see our comment in Section 4.2). The matrix $W_0 \doteq W_{0,0}$ is a modular invariant: it commutes with the generators S and T, representing $\mathrm{SL}(2, \mathbb{Z})$ in the vector space spanned by the vertices of the graph $\mathcal{A}_{\kappa-1}$. The corresponding sesquilinear form is the modular invariant partition function. The other matrices $W_{x,y}$ are associated with partition functions that are not modular invariant.

[3] If κ is the Coxeter number of G, n denotes the cardinality of the set of vertices of the diagram $A_{\kappa-1}$, i.e., $n = \kappa - 1$ for a diagram G of type ADE.

For affine models characterized by the affine Kac–Moody algebra of type $\widehat{\mathrm{su}}(N)$, the story is very similar. Here χ is still a vector of the complex vector space $\mathbb{C}^n$ but n now denotes the cardinality of the set of vertices of a graph $\mathcal{A}(G)$ generalizing the $A_{\kappa-1}$ Dynkin diagram. In the case of SU(3) for instance, the ADE diagrams are replaced by the Di Francesco–Zuber diagrams, but we can again define the bi-algebra $\mathcal{B}G$ and the two related associative algebras $\mathcal{A}(G)$ and $\mathrm{Oc}(G)$. Torus structures on these diagrams and corresponding twisted partition functions are defined as before.

1.3.2. Twisted partition functions for minimal models and their higher analogs

Minimal models. It has been known for quite a while (see for instance the book [17]) that the classification of modular invariant partition functions for minimal models, unitary or not, also follows a kind of ADE classification, in the sense that every partition function describing a minimal model can be associated with *a pair* of Dynkin diagrams[4] $(G^{(1)}, G^{(2)})$. In our set-up, this affirmation can be precisely formulated as follows: the partition function of a minimal model of type $(G^{(1)}, G^{(2)})$ can be obtained as the sesquilinear form associated with the matrix $W^{(1)}_{\underline{0},\underline{0}} \otimes W^{(2)}_{\underline{0},\underline{0}}$ where these two matrices, respectively, describe the undeformed torus structures of diagrams $G^{(1)}$ and $G^{(2)}$. It is also well known that the obtained minimal model is unitary if and only if the Coxeter numbers κ_1 and κ_2 of the two diagrams $G^{(1)}$ and $G^{(2)}$ just differ by one unit. The usual situation for minimal models corresponds therefore to the choice of the two trivial torus structures for the graphs $G^{(1)}$ and $G^{(2)}$; the possibility of replacing these two torus structures by more general ones (i.e., matrices $W^{(1)}_{\underline{0},\underline{0}}$ and $W^{(2)}_{\underline{0},\underline{0}}$ by matrices $W^{(1)}_{x_1,y_1}$ and $W^{(2)}_{x_2,y_2}$) leads to a natural classification of twisted partition functions for minimal models.

Analogs of minimal models for general Coxeter–Dynkin systems. The general case of minimal models corresponds to the choice of two graphs of type SU(2) (i.e., two arbitrary Dynkin diagrams of type ADE) but one can also replace the two ADE diagrams $G^{(1)}$ and $G^{(2)}$ by members of a higher Coxeter–Dynkin system (for example the Di Francesco–Zuber diagrams of type SU(3)) and obtain in this way similar classifications. Here the notion of “minimal model” is generalized and the corresponding partition functions, twisted or not, can be interpreted in terms of minimal models for $\mathcal{W}_n$ algebras (in particular $\mathcal{W}_3$ for the Di Francesco–Zuber diagrams).

1.4. A brief historical section

Here we make a long story short and gather only a few references. Many others can be found by looking at the quoted material. Apologies for omissions.

The study of quantum geometry of ADE graphs was, at the beginning, presented as a nice example illustrating the general theory of “paragroups” and “Ocneanu cells” [26]. This class of examples and its generalizations turned out to be very rich. Much of the theory was developed by Ocneanu himself and described (sometimes in a rather allusive way) at several meetings and conferences during the years 1995–2000 (for instance [27]). As far as we know, the first published material on this theory is [28].

[4] This property received in [23] an interpretation in the framework of the theory of local nets of von Neumann algebras.

From the physical side, many relations existing between ADE graphs and physics (models of statistical mechanics) had been already observed and investigated by Pasquier in his thesis (see [31]). A classification of modular invariant partition functions for conformal field theories of SU(2) type was obtained at the same time, i.e., at the end of the 1980s, by Cappelli et al. [6,7] in a celebrated paper. Later, Gannon (and collaborators) [20] could obtain similar results for conformal field theories based on more general affine Kac–Moody algebras.

Di Francesco and Zuber made the crucial observation [15] that the SU(3) classification could be related to a family of particular graphs (that we call the Di Francesco–Zuber diagrams), in a way similar to the relation existing between the SU(2) classification and the ADE Dynkin diagrams. Several precisions concerning this classification were brought by Ocneanu at the Bariloche School ([29], see also the lectures of Zuber and Evans [2,3] at the same school).

After the unpublished work by Ocneanu concerning the ADE themselves, it was more or less clear that the existence of modular invariant partition functions associated with these diagrams (or their generalizations) was only the tip of a theoretical iceberg. For instance, from the existence of several toric structures on ADE diagrams, it was clear that the modular invariant partition function was only describing a particular point of $\mathrm{Oc}(G)$, and that other "interesting" partition functions claiming for a physical interpretation existed in the theory. A simple algorithm allowing one to obtain the toric matrices $W_{x,\underline{0}}$ was explained in [9], following the example of E_6, and, as already mentioned, a physical interpretation of the $W_{x,y}$ in terms of conformal field theory with a boundary and defects lines was given in [37]. Using the techniques explained in [9], a systematic study of all ADE cases was performed in [11] and several interesting cases belonging to the SU(3) family were analyzed in [12]. In [19], several properties of the twisted partition functions were interpreted in terms of bimodules for Frobenius algebras. More recently (see [34] and footnote 1), it was shown how to build a lattice realization of these models.

2. Quantum geometry on ADE diagrams and their generalizations

2.1. From the classical to the quantum situation (in a nutshell)

Classical situation. Representation theory of Lie groups (SU(2), SU(3), etc.) and their subgroups can be encoded by graphs. These graphs tell us how to decompose the representations obtained by tensor multiplying irreducible representations (irreps); actually it is enough to know what happens when one tensor multiplies some irrep by the fundamental representations. Representation theory of SU(2) is encoded, in this way, by the graph A_∞ (it describes the coupling of an arbitrary spin with a spin 1/2). Representation theory of SU(3) is characterized by two generalized $\mathcal{A}_\infty$ diagrams differing only by orientation (multiplication by the fundamentals $3 = (1, 0)$ of $\bar{3} = (0, 1)$). Such a graph defines an associative algebra (the "graph algebra") which is the Grothendieck ring spanned by the irreducible characters of the group. Notice that the graph algebra of a subgroup is a module over the graph algebra of the group and that the structure constants characterizing these associative algebras, or modules, are positive integers.

Quantum situation. In the case of SU(2), truncating the diagram A_∞ leads to the usual A_n Dynkin diagrams. In the case of SU(3), truncating one of the two diagrams $\mathcal{A}_\infty$ leads to the Di Francesco–Zuber diagrams of type $\mathcal{A}$. This can be generalized to SU(N) [38]. The vector space spanned by the vertices of any $\mathcal{A}$ diagram, for a given SU(N) system, always possess a $\mathbb{Z}_N$ grading (called N-ality). For instance, in the case of the usual A_n Dynkin diagrams, vertices are either "even" or "odd". All these graphs "of type $\mathcal{A}$" have self-fusion (an associative multiplication law with positive integral structure constants), but they are not the only ones enjoying this property. The obtained graph algebras are associative and commutative algebras with a particular basis, they are denoted by the same symbol as the graph itself. For a given N (the choice of SU(N)), the first task is to determine all those diagrams which simultaneously admit N-ality, generate a module (with integral structure constants) over some associative algebras of type $\mathcal{A}$ and also admit self-fusion. The next task is to identify all those diagrams (with N-ality) which do not necessarily enjoy self-fusion, but which nevertheless generate a module (with integral structure constants) over one of the algebras defined by the previous family. A list of requirements[5] that a given diagram should obey in order to be a member of some "generalized Coxeter–Dynkin system" was given in [38], but as mentioned by Ocneanu [29] (see also [30]), this list was not complete, in the sense that a local condition of cohomological nature should also be imposed on its set of "cells"; this is not discussed here.

2.2. The classical and quantum systems of diagrams for SU(2) and SU(3)

The classical SU(2) system. Choose a finite subgroup of SU(2), i.e., one of the so-called binary polyhedral groups. The fundamental representation is again two-dimensional and the multiplication of any of its irreps by the fundamental is encoded by the corresponding diagram of tensorization, which, for the binary groups of symmetries of platonic bodies coincides with the affine exceptional Dynkin diagrams $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ (McKay [25] correspondence). The vector space generated by the set of irreps of such a subgroup is a module over the algebra generated by the set of irreps of SU(2) (reduce irreps from the group SU(2) to its subgroup and use tensor multiplication of representations). In diagrammatic parlance, we may say that affine ADE diagrams are modules over the A_∞ diagram. Irreps of a binary polyhedral group can also be tensor multiplied and decomposed into irreps (with positive integral structure constants). In other words, affine ADE diagrams have self-fusion. In particular, one of the vertices σ_j acts as the unit, we call it σ_0. For each of these diagrams, call G_1 the adjacency matrix; its highest eigenvalue β (called the Perron–Frobenius norm of the diagram) is equal to 2 in all cases and it coincides with the dimension of the fundamental representation. For a given diagram, dimensions of the irreps are given by components of the (unique) normalized eigenvector corresponding to β (it is normalized to 1 at the unit point σ_0). The table of characters S happens to be equal to the matrix of eigenvectors (properly normalized) of G_1. This is a way to express the general McKay correspondence in the case of SU(2).

[5] For instance, when looking for modules over commutative algebras associated with $\mathcal{A}$ diagrams, one should impose that they have the same generalized Coxeter numbers.

The quantum SU(2) system. Now we move to the quantum case and replace the A_∞ diagram by A_n diagrams seen as truncated A_∞ diagrams. These diagrams A_n have self-fusion. The next task is to determine those diagrams (with bi-ality) that generate modules over the A_n: we get the A, D and E diagrams. For example, E_6 is an A_{11} module, E_7 an A_{17} module, and E_8 an A_{29} module. Some of them have self-fusion (A_n, D_{eve}, E_6, E_8), others do not (D_{odd}, E_7). A diagram D_{eve} actually determines a two-parameter family of associative structures, but only two of them have structure constants which are positive integers (self-fusion); these two structures can be identified when we permute the two end points of the D_{even} fork; when such a phenomenon appears, the algebra of quantum symmetries $\mathrm{Oc}(G)$, to which we shall return later, appears to be non-commutative.

The norm of a diagram G is found to be $\beta = 2\cos(2\pi/\kappa)$, where $\kappa = n+1$ if $G = A_n$ or when G is a module over A_n. Note that $1 < \beta < 2$ (see also [22]). The quantum dimensions $q\dim_a$ of the vertices of G are obtained or defined as the components of the normalized Perron–Frobenius eigenvector (which corresponds to the eigenvalue β). For every ADE diagram, i.e., for every member G of the system that we may call "the $\mathrm{SU}(2)$ Coxeter–Dynkin system", the integer κ is called the *Coxeter number of the diagram*. All these diagrams (with or without self-fusion) can also be labeled by an integer k, called the *level of the diagram* and defined by $k = \kappa - 2$. A description of the ADE diagrams in terms of representations of quantum subgroups (a quantum analog of the McKay correspondence) was discussed by Kirillov and Ostrik [24] in the framework of modular categories.

The classical SU(3) system. Representation theory for finite subgroups of $\mathrm{SU}(3)$ is fully characterized by a family of diagrams that have self-fusion and generate modules over the graph algebra of the generalized $\mathcal{A}_\infty$ diagram of $\mathrm{SU}(3)$. All of these diagrams have a norm equal to 3.

The quantum SU(3) system. Now we move to the quantum and replace $\mathcal{A}_\infty$ by $\mathcal{A}_k$ (truncated A_∞ diagrams). These $\mathcal{A}_k$ have self-fusion. The next task is to determine those diagrams (with tri-ality) that are modules over the $\mathcal{A}_k$: we get the Di Francesco–Zuber diagrams. Some of them have self-fusion and others do not. The system contains in particular the $\mathcal{A}$ series and a finite number of "genuine exceptional" cases ($\mathcal{E}_5$, $\mathcal{E}_9$ and $\mathcal{E}_{21}$). The other diagrams of the system are obtained as orbifolds of the genuine diagrams (exceptional or not) and as twists or conjugates (sometimes both) of the genuine diagrams and of their orbifolds, see [15,29,38,39]. All of them have a norm β equal to $1 + 2\cos(2\pi/\kappa)$. Note that $2 < \beta < 3$. This again defines an integer κ called the "generalized Coxeter number" or "altitude" (like in [15]). The level k of a *diagram* belonging to this family is defined by the relation $k \doteq \kappa - 3$. The truncated $\mathcal{A}_\infty$ diagrams that we call $\mathcal{A}_k$ are of level k (see the footnote in the next subsection). Even when it exists, the determination of the graph algebra of a given diagram is not always unique; a phenomenon similar to what happens for the D_{even} diagrams (see a previous remark) occurs for instance in the case of the $\mathcal{E}_9$ diagram of the $\mathrm{SU}(3)$ system.

2.3. General notations and characteristic numbers for generalized Coxeter–Dynkin diagrams

The classical representation theory of $\mathrm{SU}(N)$ can be encoded by a set of $N-1$ diagrams (with oriented edges and infinitely many vertices) generalizing the A_∞ diagram of $\mathrm{SU}(2)$;

there is one such oriented diagram for each fundamental representation. For definiteness, we choose the basic representation of $SU(N)$; its Young tableau is given by a single box. A given system of diagrams is then labeled by an integer N, it has the same value for all diagrams of a system. For ADE Dynkin diagrams, $N = 2$, the (dual) Coxeter number of $SU(2)$. For Di Francesco–Zuber diagrams, $N = 3$, the (dual) Coxeter number of $SU(3)$. The generalized Coxeter number (altitude) of a diagram G is called κ in our paper, it can be defined directly from the norm β of G; for usual Dynkin diagrams, altitude is the usual dual Coxeter number. It is useful to define the root of unity $\mathbf{q} \doteq \exp(i\pi/\kappa)$, so that $\mathbf{q}^{2\kappa} = 1$. The level k of a given *diagram* belonging to a given system of type $SU(N)$ is defined by the relation $k \doteq \kappa - N$. More generally, one could probably define generalized Coxeter–Dynkin systems for any Lie group (the case $SU(2)$ corresponding to the ADE system), but such a theory remains to be investigated.

As we know, for a given system, members of the $\mathcal{A}$ family (call them $\mathcal{A}_k$, with k standing for the level[6]) are obtained as truncated [7] $\mathcal{A}_\infty$ diagrams. They can be related to a particular category of representations of quantum groups at roots of unity, but we shall not discuss this aspect here.

A diagram G of level k belonging to such a generalized system is always such that the vector space spanned by the set of its m vertices is a module over the member $\mathcal{A}_k$ of the $\mathcal{A}$ family with the same level[8] (the number of vertices of this corresponding diagram of type $\mathcal{A}$ will be called n, so $m = n$ when G is of type A. Notice that $n = k + 1$ for usual $A_n = \mathcal{A}_k$ Dynkin diagrams, but $n = (k + 1)(k + 2)/2$ for Di Francesco–Zuber diagrams of type $\mathcal{A}_k$.

The list of exponents $\{r\}$ of a graph G of type ADE can be defined directly from the table of eigenvalues of the adjacency matrix G_1 of G: these eigenvalues are of the form $2\,\mathrm{Cos}(r\pi/\kappa)$. For instance, in the case of E_6, from the list of eigenvalues

$$\left\{2\,\mathrm{Cos}\left(\frac{\pi}{12}\right), 2\,\mathrm{Cos}\left(\frac{4\pi}{12}\right), 2\,\mathrm{Cos}\left(\frac{5\pi}{12}\right), 2\,\mathrm{Cos}\left(\frac{7\pi}{12}\right),\right.$$
$$\left. 2\,\mathrm{Cos}\left(\frac{8\pi}{12}\right), 2\,\mathrm{Cos}\left(\frac{11\pi}{12}\right)\right\},$$

we read the exponents $\{1, 4, 5, 7, 8, 11\}$. Notice that exponents also refer to particular vertices of the corresponding diagram of type $\mathcal{A}$ with the same Coxeter number (for E_6, see the circled vertices in Fig. 3, and remember that our indices for labeling vertices are shifted by 1). The list of exponents $\{r = (r_1, r_2)\}$ of a graph G belonging to a generalized system can also be defined directly from the adjacency matrix G_1 of G. For Di Francesco–Zuber diagrams (i.e. the $SU(3)$ system), they can be read from the following general formula giving the eigenvalues of G_1 [15] $(1 + e^{2i\pi r_1/\kappa} + e^{2i\pi(r_1+r_2)/\kappa})/e^{2i\pi(2r_1+r_2)/3\kappa}$. For instance,

[6] Another favorite notation is $\mathcal{A}^{(k+N)}$, the upper index referring now to the altitude. We shall stick to the notation using level as a subscript.

[7] Truncation is made by removing the parts of the diagram with level higher than k; what we obtain is a truncated Weyl chamber ("a Weyl alcove").

[8] Warning, in the $SU(2)$ case, we have two notations for the same objects since the subindex of A_n refers usually to the number of vertices (the rank), but in this particular case, $k = n - 1$, so that $\mathcal{A}_{k=n-1} = A_n$.

the exponents of $\mathcal{E}_5$ are

$$\{(1,1),(3,3),(1,3),(4,3),(3,1),(3,4),(3,2),(1,6),(4,1),(1,4),(2,3),(6,1)\}.$$

Here again, exponents refer to particular vertices of the corresponding diagram of type $\mathcal{A}$ with the same Coxeter number and remember that indices labeling vertices are usually shifted by $(1,1)$. Exponents appear in the expression giving the corresponding modular invariant partition function (see the examples of E_6 or of $\mathcal{E}_5$ in Sections 3.1.5 and 3.2) and in the usual (or generalized) Rocha–Cariddi formulae.

2.4. Paths, essential paths, the bi-algebra $\mathcal{B}G$ and the algebra $Oc(G)$ of quantum symmetries

We then move from the geometry of the "space" G to the geometry of the paths on G, a procedure quite common in quantum physics! Paths on G generate a vector space *Paths* which comes with a grading: paths of homogeneous grade are associated with Young diagrams of $\mathrm{SU}(N)$. In the case of $\mathrm{SU}(2)$ this grading is just an integer (to be thought of as a length, a Young frame with a single row, or as a point of $A_n \equiv \mathcal{A}_{k=n-1}$).

What turns out to be most interesting is a particular vector subspace of *Paths* whose elements are called "essential paths" (we refer to [9,28], see also [10] for a definition). The space of essential paths *EssPaths* is itself graded in the same way as *Paths* and one may consider the graded algebra of endomorphisms of essential paths $\mathcal{B}G \doteq End_\sharp(EssPaths) = \oplus_{j=0,r-1} End(EssPaths^j)$.

By using the fact that paths on the chosen diagram can be concatenated, one may define *another* multiplicative (associative) structure on the vector space $\mathcal{B}G$ (see [28] for a definition). This leads to a *bi*-algebra $\mathcal{B}G$ which turns out to be semi-simple for both structures, but existence of a scalar product allows one to transmute one of the multiplications into a co-multiplication compatible with the other structure and one obtains in this way a *bi*-algebra. This bi-algebra is sometimes called, by Ocneanu "algebra of double triangles" (DTA), a terminology coming from the graphical representation of the corresponding elementary matrices by diffusion graphs or, dually, as DTA.

For these two associative laws on the same space, that we may call "composition law" and "convolution law" (or "vertical law" and "horizontal law"), there are two—usually distinct—block decompositions for $\mathcal{B}G$ (ideals corresponding to simple blocks). The first type of blocks, labeled by j, corresponds to the grading associated with points σ_j of $\mathcal{A}_k$, i.e. in the case of $\mathrm{SU}(2)$, to the lengths of the paths, and, more generally, to Young diagrams of $\mathrm{SU}(N)$; interpretation of this first block structure is therefore clear from the definition of $\mathcal{B}G$ as sum of algebras of endomorphisms. The second block decomposition can be interpreted as follows: ADE diagrams (or their $\mathrm{SU}(N)$ generalizations) may have classical symmetries, for instance, all A_n diagrams have an obvious $\mathbb{Z}_2$ symmetry; these classical symmetries (action of a finite group on vertices) can be promoted to the level of paths in an obvious way and therefore lead to particular endomorphisms of *EssPaths*; but there are more "quantum symmetries" acting on the space of essential paths than classical symmetries: irreducible quantum symmetries (call them x) are precisely associated with the blocks of $\mathcal{B}G$ for the second multiplication. We call $\mathrm{Oc}(G)$ the algebra spanned by the minimal central projectors associated with the later blocks, using the first multiplicative structure. $\mathrm{Oc}(G)$ is called the "algebra of quantum symmetries". In all cases it is an associative algebra with two generators (called "left" and "right" generators) and the Cayley graph of multiplication by these two generators, called the "Ocneanu graph of G" is also denoted by $\mathrm{Oc}(G)$. The linear span of these generators are called left and right chiral parts, and their intersection is called "ambichiral".

The number[9] of (simple) blocks of $\mathcal{B}G$ for its first multiplication, is n (the number of points of the corresponding $\mathcal{A}$ diagram); dimension of these blocks will be called d_j, $j = 1, \ldots, n$. The number of (simple) blocks of $\mathcal{B}G$ for its second multiplication, will be called s (the number of points of the corresponding Ocneanu diagram); dimension of these blocks will be called d_x, $x = 1, \ldots, s$. Existence of two block decompositions for the same underlying vector space $\mathcal{B}G$ leads obviously to the number-theoretical identity (quadratic sum rule): $\sum_{i=1,n} d_i^2 = \sum_{x=1,s} d_x^2$. In all cases explicitly studied so far, an unexpected linear sum rule also holds (in some cases one has to introduce a natural correction factor).

The direct determination of the algebra $\mathrm{Oc}(G)$, using the definition provided by Ocneanu, is not an easy task, and the corresponding graphs were first known (published) for the $\mathrm{SU}(2)$ Coxeter–Dynkin system [28]. This algebra is not always commutative. One of the purposes of [9,11], besides the calculation of the toric matrices, was actually to give an algebraic construction providing a realization of the *algebra* $\mathrm{Oc}(G)$ in terms of graph algebras associated with appropriate Dynkin diagrams. In many relatively easy cases where G admits self-fusion and is also such that $\mathrm{Oc}(G)$ is commutative, the algebra of quantum symmetries is isomorphic with $G \otimes_J G$, where J is a particular subalgebra of the graph algebra of G; the tensor product sign, taken "above J", means that we identify $au \otimes b$ and $a \otimes ub$ whenever $u \in J \subset G$. In those easy cases, and as shown in [12], the subalgebra J can be determined from the modular properties of the graph G; we shall remind the reader how this is done in a later section. Paradoxically, for Dynkin diagrams, and besides the A_n themselves, the "simple" cases happen to be those where G is an exceptional diagram equal to E_6 or E_8. We refer to [11] for a discussion of all ADE cases and [12] for a discussion of a number of cases belonging to the Di Francesco–Zuber system.

2.5. The matrices N_i, F_i, G_a, E_a, S_x and $W_{x,y}$

2.5.1. Fusion matrices: the N_i's

Fusion matrices are defined for $\mathcal{A}_k$ diagrams. They are square matrices of dimension $n \times n$ called N_i. They are associated with the vertices τ_i with $i \in \{0, \ldots, n-1\}$, and provide a faithful representation of the graph algebra. Here i is actually a multi-index referring to a Young frame of $\mathrm{SU}(N)$ and the cardinality of the indexing set is n. When the Young frame refers to a fundamental representation (only one column), this fusion matrix is the adjacency matrix of the corresponding oriented diagram. Other matrices N_i are obtained from the fundamental ones by applying the particular recurrence relation specific to $\mathrm{SU}(N)$. Example: in the case of $\mathrm{SU}(2)$, each Young diagram is an horizontal string of boxes and is characterized by its length; the matrix N_1 is the adjacency matrix of $\mathcal{A}_k = A_{n=k+1}$ and N_0

[9] This number is infinite in the classical situation (finite subgroups of Lie groups).

is the unit; the recurrence relation (coupling of spins) is

$$N_{i+1} = N_1 N_i - N_{i-1}.$$

Matrices N_i have indices (j, k) referring to vertices of $\mathcal{A}_k$. These matrices generate a (commutative) associative algebra isomorphic with the algebra of the given $\mathcal{A}$ diagram. The indices i, j runs from 0 to $n-1$ but we shall sometimes use indices $r = i+1$, $s = j+1$ running from 1 to n. In the case of SU(3), the index j labeling vertices τ_j of the $\mathcal{A}_\kappa$ diagram is a pair (j_1, j_2), with $j_1, j_2 \geq 0$ and $j_1 + j_2 \leq k$. The identity is $N_{0,0}$ and the matrix $N_{1,0}$ denotes the adjacency matrix of the (oriented) diagram G. The recurrence formula reads

$$N_{j_1,j_2} = 0 \quad \text{if} \quad j_1 < 0 \;\text{ or }\; j_2 < 0, \qquad N_{j_1,0} = N_{1,0}N_{j_1-1,0} - N_{j_1-2,1},$$
$$N_{j_1,j_2} = N_{1,0}N_{j_1-1,j_2} - N_{j_1-1,j_2-1} - N_{j_1-2,j_2+1} \quad \text{if} \quad j_2 \neq 0, \qquad N_{0,j_1} = N^{\mathrm{T}}_{j_1,0}.$$

2.5.2. Fused adjacency matrices: the F_i's

The module property (external multiplication) of the vector space associated with a diagram G, of level k and possessing m vertices, with respect to the action of the algebra $\mathcal{A}_k$ is encoded by a set of n matrices F_i, $i = 0, \ldots, n-1$, of dimension $m \times m$, sometimes called "fused graph matrices" (a somehow misleading terminology!): $\tau_i \sigma_a = \sum_b (F_i)_{ab} \sigma_b$.

If G is of type $\mathcal{A}$, we have $n = m$, $F_i = N_i$ and we are done. More generally, call F_0 the unit matrix of dimension $m \times m$, and F_1 the adjacency matrix of G. For usual ADE diagrams, each edge carries both orientations and F_1 is symmetric; for generalized diagrams, this is not so. Other matrices F_i are then obtained by imposing the same recurrence relation as for the fusion matrices. Matrices F_i have indices (a, b) referring to vertices of G; they characterize G as a module over the corresponding $\mathcal{A}$ graph. They are also in one-to-one correspondence with the minimal central projectors diagonalizing one of the two associative structures of the bi-algebra $\mathcal{B}G$, in other words they characterize the corresponding blocks and give their dimensions $d_i = \sum_{a,b} (F_i)_{a,b}$.

In the case of SU(3) diagrams, remember that indices j are pairs (j_1, j_2) and that fused adjacency matrices F_j, associated with any graph G of a given level, are determined by the same recurrence relations as for matrices $N_i = N_{j_1,j_2}$ associated with the graph $\mathcal{A}$ of the same level; only the seed is different: $F_1 \doteq G_1$, the adjacency matrix of G.

2.5.3. Graph matrices: the G_a's

The diagram G sometimes admits self-fusion. In those cases, the m linear generators σ_a of G (a runs from 0 to $m-1$) are represented by m commuting matrices G_a of dimension $m \times m$ spanning a faithful representation of the graph algebra. We call $G_0 \doteq F_0$, $G_1 \doteq F_1$ and more generally G_a the set of matrices (one for each vertex of G) representing faithfully the multiplication of vertices. Warning: with the exception of F_0 and F_1, the matrices F_i and G_a are distinct (in the case of $\mathcal{A}$ diagrams, of course, they are identical).

2.5.4. Essential matrices: the E_a's

By definition, the m essential matrices E_a are rectangular matrices of dimension $n \times m$ defined by setting,[10] for every vertex a of G, $(E_a)_{i,b} \doteq (F_i)_{a,b}$. These are rectangular matrices of dimension (n, m). Matrices E_a display "visually" the structure of essential paths emanating from a vertex a on the diagram G. One can check that, for graphs with self-fusion, $E_a = E_0 G_a$. The particular matrix E_0 is usually called "intertwiner", in the statistical physics literature [32].

As we know, vertices of the diagram G should be thought of as an analog of irreps for a subgroup of a group; the irreps of the bigger group are themselves represented by vertices of the corresponding A graph. In this analogy, the first column of each matrix F_i would describe the branching rule of τ_i with respect to the chosen subgroup (restriction mechanism). In the same way, the columns of the particular essential matrix E_0 would describe the induction mechanism: the non-zero matrix elements of the column labeled by σ_b tell us what are those representations τ_i that contain σ_b in their decomposition (for the branching $A \to G$).

2.5.5. Matrices for $Oc(G)$

Since we have a bi-algebra $\mathcal{B}G$ we have also a set of matrices S_x which characterize the blocks of the other associative structure (one for each point of the Ocneanu graph). In "simple cases", like E_6 or E_8, the matrix S_x associated with the vertex $x = a \otimes_J b$ of the Ocneanu graph is simply equal to the product $G_a G_b$. The dimension d_x of the block x is obtained by summing the matrix elements of S_x.

2.5.6. Toric matrices and generalized toric matrices: the W_x and $W_{x,y}$

We know that $\mathcal{A}_k$ acts on G, but $\mathcal{A}_k$ also acts (from both sides) on $\mathrm{Oc}(G)$. In general, $\mathrm{Oc}(G)$ is an $\mathcal{A}_k$ bimodule and the action is encoded as follows: $\tau_i x \tau_j = \sum_{y \in \mathrm{Oc}(G)} (W_{xy})^i_j y$, with $x, y \in \mathrm{Oc}(G)$ and $\tau_i, \tau_j \in \mathcal{A}_k$. In general, one obtains $s \times s = s^2$ matrices W_{xy} of dimension $s \times s$ (many of them may happen to be equal). In particular, one obtains the s matrices $W_x \doteq W_{x0}$ and the matrix $W_0 = W_{00}$ associated with the origin of the Ocneanu graph. Practically, once we have the m rectangular matrices E_a, of dimension $n \times m$, we first replace by 0 all the matrix elements of the columns labeled by vertices b that *do not* belong to the subset J of the graph G, call E_a^{red} these "reduced" matrices and obtain, for each point[11] $x = a \dot{\otimes} b$ of the Ocneanu graph $\mathrm{Oc}(G)$, a "toric matrix" $W_x = E_a (E_b^{\mathrm{red}})^{\mathrm{T}}$, of dimension $n \times n$.

We will explain in Section 3.1.5 how to generalize the previous method to obtain all the toric matrices $W_{x,y}$ ("first algorithm"). Actually, the $W_{x,y}$ can also be obtained from the W_x, determined as above, by working out the multiplication table of $\mathrm{Oc}(G)$ (this is our "second algorithm"). All we have to do is to decompose the product $x \times y$ on the basis generators: if $x \cdot y = \Sigma_z C^z_{x,y} z$ with $x, y, z \in \mathrm{Oc}(G)$ then $W_{x,y} = \Sigma_z C^z_{x,y} W_{0,z}$. This can be seen as a compatibility equation; indeed, the action of $\mathcal{A}_k$ is central, so $\tau_i \cdot x \cdot \tau_j = x \cdot \tau_i \cdot \underline{0} \cdot \tau_j$

[10] The reader should be cautious about the meaning of indices: our indices i or a refer to actual vertices of the graphs but the numbers chosen for labeling matrix rows and columns depend on some arbitrary ordering on these sets of vertices. Moreover, our labels i and a start from 0, not from 1.

[11] In some cases, x may be a linear combination of such elements.

implies

$$\Sigma_y (W_{xy})^i_j y = x \cdot (\Sigma_z (W_{0z})^i_j z) = \Sigma_z (W_{0z})^i_j x \cdot z = \Sigma_z (W_{0z})^i_j \Sigma_y C^y_{x,z} y$$
$$= \Sigma_y (\Sigma_z C^y_{x,z} (W_{0z})^i_j) y.$$

Notice that linearity of this relation implies in particular $W_{xy,0} = W_{x,y}$. Moreover, when $\mathrm{Oc}(G)$ is commutative, i.e. $xy = yx$, we have $W_{x,y} = W_{y,x}$ (but the later equality does not imply the former).

From the toric matrices W_{xy} describing the bimodule structure of $\mathrm{Oc}(G)$, one obtains the corresponding twisted partition functions as sesquilinear forms in the complex vector space $\mathbb{C}^s$. Introducing a basis χ of vectors (χ_j), usually interpreted as characters, we write

$$Z_{x,y} = \bar{\chi} W_{x,y} \chi$$

and $Z_x = Z_{x,\underline{0}}$. The modular invariant partition function is $Z_{\underline{0}}$ with $\underline{0} = 0 \dot{\otimes} 0$. The example of E_6 is discussed in Section 3.1.

2.6. Modular aspects: S, T and $SL(2, \mathbb{Z})$

2.6.1. The S operator

Any finite subgroup of $\mathrm{SU}(2)$ can be associated with an affine ADE graph, in such a way that the normalized Perron–Frobenius vector of the graph gives the list of dimensions for irreps of the finite subgroup. This observation, known as McKay correspondence, was later generalized by observing that the whole table of characters of a finite subgroup of $\mathrm{SU}(2)$ can be identified with the list of eigenvectors (properly normalized) of the adjacency matrix of the corresponding affine Dynkin diagram (generalized McKay correspondence). For any finite group, not necessarily a subgroup of $\mathrm{SU}(2)$, the commutative and associative algebra generated by irreducible characters (multiplication of representations) can be realized by a set of commuting matrices (the analog of our matrices G_a) and the table of characters can be reconstructed, without using the notion of conjugacy classes, by diagonalizing simultaneously this set of (commuting) matrices: the character table S is a properly normalized diagonalizing matrix. The following “quantum construction” is analogous.

In the quantum case (i.e. diagrams ADE), there is no group, there are no conjugacy classes and no table of characters. Nevertheless, there is an adjacency matrix for the chosen diagram. The matrix S that we are looking for is precisely the quantum analog of the table of characters, and is obtained, for each level k as the (properly normalized) table of eigenvectors for the adjacency matrix of the diagram $\mathcal{A}_k$. The bonus in the quantum situation is that one can interpret S as one of the generators of the modular group in a particular representation; this representation of $\mathrm{SL}(2, \mathbb{Z})$ appeared in a work by Hurwitz [21] about a century ago. S, interpreted as a quantum table of characters (or a “quantum Fourier transform”) implements therefore a quantum analog of the McKay correspondence. For illustration, the modular matrix S for the A_{11} diagram is determined in this way in Section 3.1.8. The general expression for $S = s$, in the case of the $\mathrm{SU}(2)$ system, with $\kappa = k + 2$, is

$$S_{ij} = \sqrt{\frac{2}{\kappa}} \sin\left(\pi \frac{(i+1)(j+1)}{\kappa}\right) \quad \text{for } 0 \le i, j \le \kappa - 2.$$

2.6.2. $SL(2, \mathbb{Z})$

A projective representation of $\mathrm{SL}(2, \mathbb{Z})$ can be defined with two matrices s and t and a phase ζ which are such that $(st)^3 = \zeta^3 s^2$, $s^2 = C$, $Ct = tC$ and $C^2 = 1$. The matrix C is called "conjugation matrix" and t the "modular twist". Such representations of the modular group can be obtained on the space generated by the simple objects in any braided modular category [1]. The general formula for the modular phase is $\zeta = \mathrm{e}^{2\mathrm{i}\pi c/24}$ with $c = (\kappa - N)d/\kappa$. In the present context, i.e. generalized Coxeter–Dynkin diagrams of type $\mathrm{SU}(N)$, κ is the altitude (generalized Coxeter), $\kappa - N = k$ is the level and $d = \dim \mathrm{SU}(N)$. Therefore, $c = 3k/(k + 2)$ for SU(2) and $c = 8k/(k + 3)$ for SU(3). The modular phase ζ is then equal to $\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{-\mathrm{i}\pi/2\kappa}$ for an ADE diagram and to $\mathrm{e}^{2\mathrm{i}\pi/3}\,\mathrm{e}^{-2\mathrm{i}\pi/\kappa}$ for a Di Francesco–Zuber diagram. We use modular generators S, T normalized as follows: $S = s$ and $T = t/\zeta$. The $\mathrm{SL}(2, Z)$ relations then read $(ST)^3 = S^2$, $S^2 = 1$.

2.6.3. The T operator

In the framework of modular categories, and for a Lie algebra $\mathcal{G}$, a general expression for the modular twist is $t_{ij} = \delta_{ij}\mathbf{q}^{\langle\langle j, j+2\rho\rangle\rangle}$, where $\mathbf{q} = \mathrm{e}^{\mathrm{i}\pi/\kappa}$, ρ is half the sum of positive roots, i, j are elements of the weight lattice characterizing the representation τ_i and τ_j; moreover, $\langle\langle\cdot, \cdot\rangle\rangle$ is an invariant bilinear form on $\mathcal{G}$ normalized by $\langle\langle\alpha, \alpha\rangle\rangle = 2$ for a short root α. For SU(2), with $i, j = 0, \ldots, \kappa - 2$, the modular twist is $t_{ij} = \mathrm{e}^{(\mathrm{i}\pi/2\kappa)j(j+2)}\delta_{ij}$. Its logarithm is proportional to the Casimir operator: j is related with the (would be) spin ℓ by $j + 1 = 2\ell + 1$, therefore $(\mathrm{i}\pi/2\kappa)j(j + 2) = (2\mathrm{i}\pi/\kappa)\ell(\ell + 1)$. With our normalization, the modular generator T is therefore

$$T_{ij} = \exp\left[2\mathrm{i}\pi\left(\frac{(j+1)^2}{4\kappa} - \frac{1}{8}\right)\right]\delta_{ij}.$$

The expression $[((j+1)^2/4\kappa) - (1/8)]$ is the "modular anomaly", and it is convenient to call "modular exponent" the quantity $\hat{T} = (j+1)^2 \bmod 4\kappa$ (we could as well use $j(j+2) \bmod 4\kappa$ or any other expression differing by a constant shift).

In the case of SU(3), the action of the modular matrix T on vertices $\tau_j \equiv \tau_{(j_1, j_2)}$ of $\mathcal{A}_k$ is also diagonal and given by

$$(T)_{ij} = e_\kappa[-(i_1 + 1)^2 - (i_1 + 1)\cdot(i_2 + 1) - (i_2 + 1)^2 + \kappa]\delta_{ij},$$

where $i \doteq (i_1, i_2)$, $j \doteq (j_1, j_2)$, $e_\kappa[x] \doteq \exp(-2\mathrm{i}\pi x/3\kappa)$, and $\kappa = k + 3$. We call "modular exponent" the quantity $\hat{T} = [-(i_1 + 1)^2 - (i_1 + 1)\cdot(i_2 + 1) - (i_2 + 1)^2 + \kappa] \bmod 3\kappa$.

2.6.4. Modular invariance

Modular invariance of the partition function Z_{00} can be proven either by checking that it is invariant when we replace the modular parameter τ by $\tau + 1$ or $-1/\tau$ in the characters χ_r (these functions are generalized Jacobi's theta functions) or, much more simply, by showing that the matrix W_0 commutes with the generators S and T of the modular group in this representation.

It can be checked, from the explicit expressions of S and T in the SU(2) case, that, $T^{8\kappa} = 1$ when κ is odd and $T^{4\kappa} = 1$ when κ is even. This, by itself, is not enough to imply the following property, which is nevertheless true, and was proven more than 100 years ago [21]: the Hurwitz–Verlinde representation of $\mathrm{SL}(2,\mathbb{Z})$ factorizes over the finite group $\mathrm{SL}(2,\mathbb{Z}/8\kappa\mathbb{Z})$ when κ is odd, and over $\mathrm{SL}(2,\mathbb{Z}/4\kappa\mathbb{Z})$ when κ is even. For instance, $T^{40}=1$ for A_4 ($40=8\times 5$), but $T^{48}=1$ for A_{11} ($48=4\times 12$).

2.6.5. Determination of $Oc(G)$ from the modular properties of the diagram G

In general, an Ocneanu cell system is defined by four graphs—two horizontal and two vertical—satisfying a number of matching properties (see [18,26]). Particularly interesting cell systems are obtained when one chooses the two horizontal graphs as given by two Dynkin diagrams with the same Coxeter number. In the present situation, these two are given by the same Dynkin diagram G (we write "Dynkin" but this graph can be a member of an higher system). A priori, the determination of $\mathrm{Oc}(G)$ results from the study of the block structure of $\mathcal{B}G$ for its convolution law. This, in turn, requires the determination of the values of all Ocneanu cells for the graph system of type (G,G), a task that may involve rather long calculations ... but if our only purpose is to determine $\mathrm{Oc}(G)$, it is simpler to find a short cut. One possibility is to use the fact that we already know, in many cases, the expression of the modular invariant (as calculated by Cappelli et al. [6,7] for $\mathrm{SU}(2)$ and Gannon [20] for $\mathrm{SU}(3)$); such a technique was apparently followed by Ocneanu himself in his determination of the irreducible quantum symmetries x, also called "irreducible connections", associated with a given diagram. However, if we do not want to use this a priori knowledge, there is another technique, which uses modular properties of the diagram; this was one of the purposes of the article [12].

The $\mathcal{A}$ series is always modular: one can define a representation[12] of $\mathrm{SL}(2,\mathbb{Z})$ on the vector space of every diagram of this class and the operator T is diagonal on the vertices. Take now G some member of a generalized Dynkin–Coxeter system, and call $A=\mathcal{A}(G)$ the corresponding member of the $\mathcal{A}$ series (same Coxeter number or altitude). Being a module over the algebra of A, there are induction–restriction maps between G and A. These maps are described by the essential matrices E_a or by matrices F_i (see Section 2.5.4 and [9,11]). One can try to define an action of $\mathrm{SL}(2,\mathbb{Z})$ on the vector space of G in a way that should be compatible with those maps, but this is not necessarily possible. In plain terms: suppose that the vertex σ of G appears both in the branching rules (restriction map from A to G) of vertices τ_p and τ_q of A; one could think of defining the value of the modular generator T on σ either as $T(\tau_p)$ or as $T(\tau_q)$, but this is ambiguous, unless these two values are equal. In general, there is only a subset J of the vertices of G for which T can be defined: a vertex σ will belong to this subset whenever T is constant along the vertices of A whose restriction to G contains σ. The knowledge of this set J allows one, in the "simple cases", to determine $\mathrm{Oc}(G)$, the algebra of quantum symmetries of G: the set J generates a particular subalgebra of G and one finds $\mathrm{Oc}(G)=G\otimes_J G$.

Results for the ADE systems. For diagrams of type $\mathcal{A}$, the subalgebra J coincides with the algebra of the diagram itself, so that $\mathrm{Oc}(\mathcal{A})$ is isomorphic with $\mathcal{A}$. For E_6, the subalgebra J, isomorphic with A_3 is generated by the three extremal points, and $\mathrm{Oc}(E_6)=E_6\otimes_{A_3}E_6$ has dimension 12 (notice that $\kappa=12$, as well, but this is an accident). For E_8, the subalgebra J, isomorphic with A_2 is generated by the two extremal points of the long branches, and $\mathrm{Oc}(E_8)=E_8\otimes_{A_2}E_8$ has dimension 32 (notice that $\kappa=30$). The other cases are more

[12] Actually this representation factors to a finite group.

difficult to analyze: $\mathrm{Oc}(E_7) = D_{10} \otimes_\rho D_{10}$, where the exceptional twist ρ can be determined from the modular properties (with respect to T) of the A_{17} diagram; its dimension is 17. The algebra of quantum symmetries for a D_{odd} diagram can be written as a quotient (using an identification map ρ) of the tensor square of the associated algebra of type $\mathcal{A}$ (for instance, $\mathrm{Oc}(D_5) = A_7 \otimes_{\rho(A_7)} A_7$); the Ocneanu graph of D_{2n+1} has $4n-1$ vertices. In some respect, the determination of Ocneanu graphs for D_{even} diagrams is more difficult; indeed, the algebra of quantum symmetries, in this case, is not commutative. We sketch its construction because the result will be used later in our study of the twisted partition functions for the Potts model. Starting from D_{2n}, one first obtains the induction–restriction rules with respect to the corresponding A diagram with the same norm (A_{4n-3}) by calculating the essential matrices; from these rules and from the expression of the modular operator T on A_{4n-3}, one determines the set J. One finds that $\mathrm{Oc}(D_{2n})$ consists of two separate components. The first is given by $D_{2n}^{\mathrm{trunc}} \otimes_{J'} D_{2n}^{\mathrm{trunc}}$, where D_{2n}^{trunc} is the vector space corresponding to the subdiagram spanned by $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{2n-3}\}$, obtained by removing the fork, and $J' = \{\sigma_0, \sigma_2, \ldots, \sigma_{2n-4}\}$ is the corresponding truncated subset[13] of J. The second component is a non-commutative 2×2 matrix algebra reflecting the indistinguishability of σ_{2n-2} and σ'_{2n-2}. Ambichiral points are associated with the $n+1$ vertices of J (i.e., $n-1$ for the linear branch and 2 for the fork); the Ocneanu graph of D_{2n} has $[(2n-2)(2n-2)/(n-1)]+4 = 4n$ vertices.

Results for the SU(3) system: there is no complete treatment in the available literature, but several examples have been worked out in [12]. Because we shall use it later (see Section 5.2.2) in our study of twisted minimal models of type $\mathcal{W}_3$, we just mention that the Ocneanu graph of the exceptional $\mathcal{E}_5$ diagram has 24 points; both left and right chiral subgraphs have 12 points; the ambichiral subalgebra is of dimension 6 and the supplementary subspace has also dimension 6.

2.7. *Characters for affine models*

Strictly speaking, we do not need to use characters in this paper since modular properties of the partition functions are to be discussed in terms of commutation relations between the toric matrices and the S, T generators of $\mathrm{SL}(2, \mathbb{Z})$. However, for completeness sake, and for the reader who wants to check explicitly the results in terms of invariance, or non-invariance, with respect to transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ (or $\tau \to \tau + N$, for T^N), we remind the definitions of the characters as functions of τ, for affine models. Here τ is a point in the upper-half plane and we set $q \doteq e^{2i\pi\tau}$. These characters provide a basis of the vector space $\mathbb{C}^n$, for the defining representation (matrices N_i) of the graph algebra of diagrams of type $\mathcal{A}$. In the case of the SU(2) system, $k = \kappa - 2$ denotes the level, and for each vertex $j = 0, \ldots, k$ of a diagram $A_{k+1} = \mathcal{A}_k$, we set $r = j + 1 \equiv 2\ell + 1$ and define

$$\xi_j^{(k)}(q) = \frac{\sum_{t=-\infty}^{\infty} (2\kappa t + r) q^{(2\kappa t + r)^2/4\kappa}}{\eta(\tau)}.$$

[13] We choose the natural order to label vertices σ_a of D_{2n}.

A closed form, for this expression, is

$$\xi_j^{(k)}(q) = q^{(1+j)^2/4(2+k)} \times \frac{((1+j)(1+\theta(3,(1+j)\tau, q^{2+k})) - \mathrm{i}(2+k)\theta'(3,(1+j)\tau, q^{2+k}))}{\eta(\tau)^3},$$

where $\eta(\tau)$ is the Dedekind eta function, $\theta[3, u, v]$ is the third elliptic Jacobi theta function, and $\theta'[3, u, v]$ is its first derivative with respect to u. More explicitly, these characters read

$$\xi_j^{(k)}(q) = q^{-(1/8)+((1+j)^2/4(2+k))} \frac{\sum_{t=-\infty}^{+\infty}(j+1+2t(k+2))q^{t(j+1+t(k+2))}}{\sum_{t=-\infty}^{+\infty}(1+4t)q^{t(1+2t)}}.$$

When $\tau \to \mathrm{i}\infty$, then $\xi_j^{(k)}(q) \simeq (j+1)q^{-(1/8)+h}$ with $h = (j+1)^2/4\kappa$. The power of q is negative when $r = j+1 < \sqrt{\kappa/2}$. It is often convenient to use expressions that are valid in a neighborhood of infinity, for instance:

Graph A_1:

$$\xi_0^{(0)}(q) = 1.$$

Graph A_2:

$$\xi_0^{(1)}(q) = q^{-1/24}(1 + 3q + 4q^2 + 7q^3 + 13q^4 + 19q^5 + 29q^6 + \cdots),$$

$$\xi_1^{(1)}(q) = q^{5/24}(2 + 2q + 6q^2 + 8q^3 + 14q^4 + 20q^5 + 34q^6 + \cdots).$$

Graph A_3:

$$\xi_0^{(2)}(q) = q^{-1/16}(1 + 3q + 9q^2 + 15q^3 + 30q^4 + 54q^5 + 94q^6 + \cdots),$$

$$\xi_1^{(2)}(q) = q^{1/8}(2 + 6q + 12q^2 + 26q^3 + 48q^4 + 84q^5 + 146q^6 + \cdots),$$

$$\xi_2^{(2)}(q) = q^{7/16}(3 + 4q + 12q^2 + 21q^3 + 43q^4 + 69q^5 + 123q^6 + \cdots).$$

SU(3) characters have similar expressions, but indices j refer then to a Young frame with two rows. Because of the two existing conventions, (i, j) or (r, s), for the label of the origin (0 or 1) it is convenient to set $\chi_1 \equiv \xi_0$, $\chi_2 \equiv \xi_1$, etc.,

$$\chi_{j+1}^{(k)} \doteq \xi_j^{(k)}.$$

3. Torus structures for affine models

3.1. Example of an affine model: the E_6 case

Toric matrices W_{x0} have been determined for all ADE cases and a few others. Since we shall need them later, we summarize the situation for E_6. We also present, in this

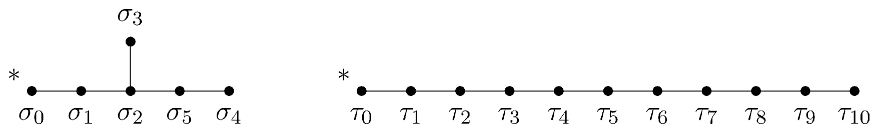
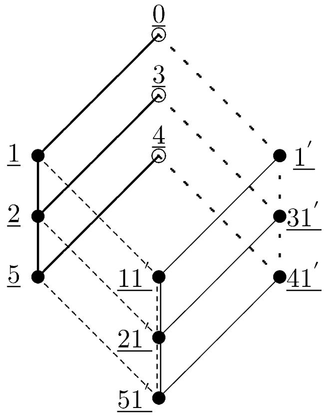


Fig. 1. The E_6 and A_{11} Dynkin diagrams.



$$W_0 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Fig. 2. The E_6 Ocneanu graph and its modular invariant.

case, several results that were not available before: the full multiplication table of Oc($E6$), the determination of the frustrated partition functions with two twists, and a discussion of modular properties of these functions. We also show how to display the expression of these partition functions in a compact way, by using induction–restriction rules for the pair (A_{11}, E_6).

3.1.1. The E_6 diagram and its Ocneanu graph (summary)

Fig. 1 displays E_6 and the related diagram A_{11}. Vertices σ_a of E_6 are labeled 0, 1, 2, 5, 4, 3 as shown in the picture. A_{11} acts on E_6, hence A_{11} also acts from the left and from the right on the Ocneanu algebra[14] of quantum symmetries which can be shown to be equal [9,11] to $\mathrm{Oc}(E_6) = E_6 \otimes_{A_3} E_6$. It has dimension 12.

The bimodule structure of $\mathrm{Oc}(E_6)$ over A_{11} is encoded by $12 \times 12 = 144$ matrices W_{xy} of dimension 11×11 (as we shall see, many of them are equal). In particular, one obtains the 12 matrices $W_x \doteq W_{x0}$, one for each point of the Ocneanu graph, and the matrix $W_0 \doteq 0 \dot{\otimes}_{A_3} 0$ associated with the origin $\underline{0}$. Fig. 2 displays the Ocneanu graph and the matrix

[14] This tensor product is taken above the subalgebra A_3 generated by vertices 0, 4, 3, so that $a \dot{\otimes} ub = au \dot{\otimes} b$ when $u \in A_3$.

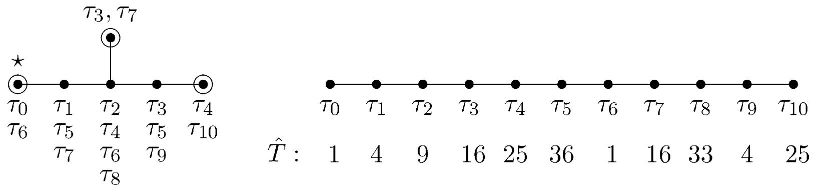


Fig. 3. The $E_6 \hookleftarrow A_{11}$ induction graph relative to vertex σ_0, and the values of $\hat{T}$ on irreps of A_{11}.

W_0. Continuous and dashed lines on this graph describe, respectively, the multiplications by the left and right chiral generators $\underline{1} = 1\dot{\otimes}0$, $\underline{1}' = 0\dot{\otimes}1$. We use the notations $\underline{a} = a\dot{\otimes}0$, $\underline{a}' = 0\dot{\otimes}a$ and $\underline{ab}' \equiv \underline{ab'} \equiv a\dot{\otimes}b$. There are many identities hidden in this graph, like for instance $\underline{31}' = \underline{2}'$; to see them, the reader should work out for himself the multiplication table of the graph algebra of E_6 or refer to [9] or [11].

3.1.2. *Induction–restriction mechanism and* $Oc(G)$

From the diagram E_6 alone, we can determine the six essential matrices E_a of dimension $(11, 6)$, as explained before. Rows of E_0 give the restriction (branching) rules $A_{11} \to E_6$ and columns give the induction rules. Induction rules are displayed in Fig. 3. We also give the values of the modular exponent $\hat{T}$ for the vertices τ_i's of A_{11}.

We notice that the value of the modular matrix T on τ_0 and τ_6 is the same (also for τ_3 and τ_7, and for τ_4 and τ_{10}). This allows one to assign a fixed value of T to three particular vertices of E_6. For every other point of the E_6 graph, the value of T that would be inherited from the A_n graph by this induction mechanism is not uniquely determined. These elements $\{\sigma_0, \sigma_3, \sigma_4\}$ span the subalgebra J isomorphic with the graph algebra of A_3; it admits an invariant supplement in the graph algebra of E_6. Using this determination of J, as explained in Section 2.6.5 (or [12]), the algebra $\mathrm{Oc}(E_6)$ is found to be equal to $E_6 \otimes_{A_3} E_6$ (Fig. 4).

3.1.3. *Linear and quadratic sum rules*

Dimensions of the 11 blocks d_i are equal to $(6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6)$. Dimension of the 12 blocks d_x are equal to $(6, 8, 6, 10, 14, 10, 10, 14, 10, 20, 28, 20)$. The quadratic sum rule reads: $\sum_i d_i^2 = \sum_x d_x^2 = 2512$. The linear sum rule also holds: $\sum_i d_i = \sum_x d_x = 156$.

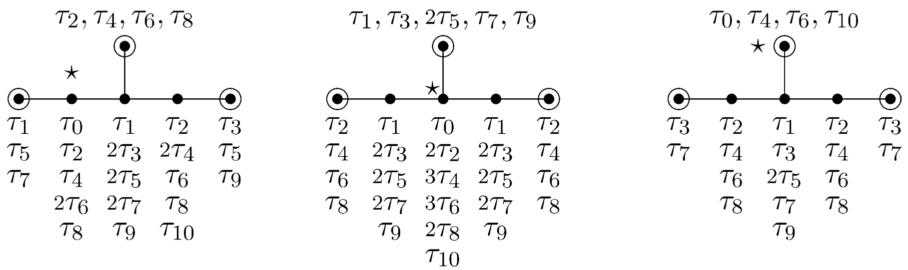


Fig. 4. The $E_6 \hookleftarrow A_{11}$ induction graphs relative to vertices σ_1, σ_2 and σ_3.

There are also quantum sum rules (mass relations): define $\mathrm{o}(G) \doteq \sum_{a\in G} q\dim_a^2$, where $q\dim_a$ are the quantum dimensions of the vertices a of G (for example $\mathrm{o}(E_6) = 4(3+\sqrt{3})$, $\mathrm{o}(A_{11}) = 24(2+\sqrt{3})$, $\mathrm{o}(A_3) = 1+(\sqrt{2})^2 = 4$); then, if G is a module over $\mathcal{A}_k$ (for some k) and when $\mathrm{Oc}(G) = G \otimes_J G$, one can check that $\mathrm{o}(\mathrm{Oc}(G))$ defined as $\mathrm{o}(G) \times \mathrm{o}(G)/\mathrm{o}(J)$ is equal to $\mathrm{o}(\mathcal{A}_k)$, for instance $\mathrm{o}(E_6) \times \mathrm{o}(E_6)/\mathrm{o}(A_3) = \mathrm{o}(A_{11})$; we do not know any general formal proof of these quantum relations.

3.1.4. Toric matrices $W_{x\underline{0}}$ and frustrated functions with one twist (results)

The toric matrices $W_{x\underline{0}}$ calculated as explained in Section 2.5.6 were explicitly listed in [9] and the corresponding partition functions $Z_{x\underline{0}}$ also appear in [11]. We recall the results:[15]

$$
\begin{array}{cc}
W_{00} & W_{11} \\
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
&
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix},
\end{array}
$$

$$
\begin{array}{cc}
W_{30} & W_{21} \\
\begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{pmatrix}
&
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix},
\end{array}
$$

[15] For $x = a\dot{\otimes}b$, we simply call $W_{ab} \doteq W_{a\dot{\otimes}b,\underline{0}}$.

$$\overset{W_{40}}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}
\quad
\overset{W_{51}}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&2&\cdot&1&\cdot&1&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}},$$

$$\overset{W_{10}}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}
\quad
\overset{W_{01}}{\begin{pmatrix}
\cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix}},$$

$$\overset{W_{20}}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}
\quad
\overset{W_{02}=W_{31}}{\begin{pmatrix}
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&2&\cdot&1&\cdot&1&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&2&\cdot&1&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot
\end{pmatrix}},$$

$$
\begin{array}{cc}
W_{50} & W_{05} = W_{41} \\
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
&
\begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot
\end{pmatrix}
\end{array}.
$$

We shall come back to these toric matrices with a single twist at the end of the next section and write the corresponding partition functions in a compact way.

3.1.5. Toric matrices W_{xy} from induction graphs

Here we give a first algorithm allowing a simple determination of all the W_{xy}. For simplicity, we choose to carry this discussion in the case of the E_6 graph. In the case of toric matrices with a single twist, this algorithm was described in [9,11], it is described here in the case of arbitrary toric matrices (two twists). It therefore generalizes the method of the previous section and uses the data given by essential matrices (E_6/A_{11} induction rules). Another algorithm for the determination of the W_{xy}, using the multiplication table of the algebra of quantum symmetries, will be described later.

Call $\mathcal{V}_{034}[a]$ the $(11, 3)$ rectangular matrix describing the $E_6 \leftarrow A_{11}$ induction graph relative to the vertex σ_a and restricted to vertices $\sigma_0, \sigma_3, \sigma_4$ of E_6 (spanning the subalgebra J isomorphic with A_3). Call $\mathcal{V}_{125}[a]$ the analogous $(11, 3)$ matrix relative to the same vertex σ_a but obtained by restriction to vertices $\sigma_1, \sigma_2, \sigma_5$ (spanning a supplement of J). Both matrices (and induction graphs) can be obtained from the $(11, 6)$ essential matrix E_a by keeping only the columns labeled by 0, 3, 4 (respectively, those labeled by 1, 2, 5). The induction graph relative to vertex σ_0 was given in Fig. 3; we also give the induction graph relative to vertices σ_1, σ_2 and σ_3 in Fig. 4; graphs relative to σ_5 and σ_4 are obtained from those relative to σ_1 and σ_0 by $\mathbb{Z}_2$ symmetry.

We need to use the three graph matrices of A_3, obviously given by

$$
W_0(A_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W_1(A_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad W_2(A_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Since A_3 is a member of the A series, graph matrices, essential matrices and toric matrices of type W_{x0} are equal. Remember that, in the isomorphism $J \simeq A_3$, indices 0, 1, 2 of A_3 are associated with indices 0, 3, 4 of E_6. Toric matrices of E_6 are of dimension $(11, 11)$; they can be written as products of matrices of dimension $(11, 3)(3, 3)(3, 11)$, where the $(3, 3)$

matrices are the toric matrices of A_3 and where the rectangular matrices of dimensions $(11, 3)$ or $(3, 11)$ give the induction/restriction rules from A_{11} to A_3.

The set of toric matrices W_{xy} with twists $x = a\dot{\otimes}b$ and $y = c\dot{\otimes}d$, written $W_{ab,cd}$ is

$$\{W_{x,cd}\}_{x\in \mathrm{Oc}(E_6)} = \left\{ \begin{array}{c} \mathcal{V}_{034}[c]W_i(A_3)\mathcal{V}^{\mathrm{T}}_{034}[d] \\ \mathcal{V}_{125}[c]W_i(A_3)\mathcal{V}^{\mathrm{T}}_{034}[d],\ \mathcal{V}_{034}[c]W_i(A_3)\mathcal{V}^{\mathrm{T}}_{125}[d] \\ \mathcal{V}_{125}[c]W_i(A_3)\mathcal{V}^{\mathrm{T}}_{125}[d] \end{array} \right\}_{i=1,2,3}.$$

The above table exhibits, on purpose, a one-to-one correspondence with the drawing of the Ocneanu graph (Fig. 2), with ambichiral generators on the first line, left and right chiral generators on the second line and supplementary generators on the third line. Moreover, the correspondence with i indices of A_3 runs from top to bottom on each vertical of Fig. 2. For instance, $x = \underline{21}' = 2\dot{\otimes}1$, is the second supplementary generator, so that a matrix $W_{21,cd}$ is equal to $\mathcal{V}_{125}[c]W_1(A_3)\mathcal{V}^{\mathrm{T}}_{125}[d]$.

Introducing the "adapted vectors" $w[a]\doteq\mathcal{V}_{034}[a]\cdot\chi$ and $v[a]\doteq\mathcal{V}_{125}[a]\cdot\chi$, where χ is the vector of characters,[16] we write the partition functions Z_{xy} associated with matrices W_{xy} as

$$\{Z_{x,cd}\}_{x\in \mathrm{Oc}(E_6)} = \left\{ \begin{array}{c} (\bar\chi\cdot w[c])W_i(A_3)(w[d]^{\mathrm{T}}\cdot\chi) \\ (\bar\chi\cdot v[c])W_i(A_3)(w[d]^{\mathrm{T}}\cdot\chi),\ (\bar\chi\cdot w[c])W_i(A_3)(v[d]^{\mathrm{T}}\cdot\chi) \\ (\bar\chi\cdot v[c])W_i(A_3)(v[d]^{\mathrm{T}}\cdot\chi) \end{array} \right\}_{i=1,2,3}.$$

For instance, $Z_{21,cd} = (\bar\chi\cdot v[c])W_1(A_3)(v[d]^{\mathrm{T}}\cdot\chi)$, and $Z_{21}\doteq Z_{21,00} = (\bar\chi\cdot v[0])W_1(A_3)$ $(v[0]^{\mathrm{T}}\cdot\chi)$.

Altogether, we have six adapted vectors $v[a]$ and six adapted vectors $w[a]$, all of them have three components.[17] The use of these two adapted vectors $w[a]$ and $v[a]$ associated with induction rules for the vertex a allows one to write all the results for W_{xy} (or Z_{xy}) in a very compact way. Let us now rewrite the partition functions with one twist, already obtained in the last section (matrices W_x) in terms of these adapted vectors.

Adapted vectors for the vertex 0 (see Fig. 3):

$$w_1\doteq w_1[0] = \chi_1+\chi_7,\qquad v_1\doteq v_1[0] = \chi_2+\chi_6+\chi_8,\qquad w_2\doteq w_2[0] = \chi_4+\chi_8,$$
$$v_2\doteq v_2[0] = \chi_3+\chi_5+\chi_7+\chi_9,\qquad w_3\doteq w_3[0] = \chi_{11}+\chi_5,$$
$$v_3\doteq v_3[0] = \chi_4+\chi_6+\chi_{10}.$$

With $v = v[0] = \{v_1, v_2, v_3\}$ and $w = w[0] = \{w_1, w_2, w_3\}$, the twisted partition functions of type Z_{x0} read

$$Z_{00} = \bar w W_0(A_3)w,\qquad Z_{30} = \bar w W_1(A_3)w,\qquad Z_{40} = \bar w W_2(A_3)w,$$
$$Z_{10} = \bar w W_0(A_3)v,\qquad Z_{01} = \bar v(W_0(A_3))w,\qquad Z_{20} = \bar w W_1(A_3)v,$$

[16] We denote the 11 characters of A_{11} by $\chi_{j+1}\equiv\xi_j$, with $j+1 = 1,\ldots,11$, dropping the upper index k which is always equal to 10 in this case.

[17] When studying the E_8 graph and its induction pattern relative to A_{29}, J happens to be two-dimensional, so the $w[a]$ will have two components and the $v[a]$ will have six.

$$Z_{31} = \bar{v}(W_1(A_3))w, \qquad Z_{50} = \bar{w}W_2(A_3)v, \qquad Z_{41} = \bar{v}(W_2(A_3))w,$$
$$Z_{11} = \bar{v}W_0(A_3)v, \qquad Z_{21} = \bar{v}W_1(A_3)v, \qquad Z_{51} = \bar{v}W_2(A_3)v.$$

The first entry, Z_{00}, is the usual (modular invariant) partition function. Explicitly, we rewrite the 12 partition functions $Z_{ab} \doteq Z_{ab,00}$ (ambichiral, left and right chiral, and supplementary) in terms of these six linear combinations of characters v and w as follows:

$$Z_{00}(q) = |w_1|^2 + |w_2|^2 + |w_3|^2, \qquad Z_{30}(q) = (\bar{w}_1 + \bar{w}_3)w_2 + \bar{w}_2(w_1 + w_3),$$
$$Z_{40}(q) = \bar{w}_1 w_3 + \bar{w}_3 w_1 + |w_2|^2,$$

$$Z_{10}(q) = \bar{v}_3 w_3 + \bar{v}_1 w_1 + \bar{v}_2 w_2,$$
$$Z_{01} = \bar{Z}_{10}, \qquad Z_{20}(q) = (\bar{v}_1 + \bar{v}_3)w_2 + \bar{v}_2(w_1 + w_3), \qquad Z_{02} = \bar{Z}_{20},$$
$$Z_{50}(q) = \bar{v}_3 w_1 + \bar{v}_1 w_3 + \bar{v}_2 w_2, \qquad Z_{05} = \bar{Z}_{50},$$

$$Z_{11}(q) = |v_1|^2 + |v_2|^2 + |v_3|^2, \qquad Z_{21}(q) = (\bar{v}_1 + \bar{v}_3)v_2 + \bar{v}_2(v_1 + v_3),$$
$$Z_{51}(q) = \bar{v}_1 v_3 + \bar{v}_3 v_1 + |v_2|^2.$$

The table of twisted partition functions Z_{xy} for all ADE models appearing at the end of Ref. [12] could be greatly simplified by using this compact reformulation.

3.1.6. The multiplication table for Oc(E_6)

As we know, the Ocneanu graph encodes the result of multiplication of basis elements of Oc(G) by the two chiral left and right generators. Determination of the full table of multiplication of Oc(G) can then be obtained in a straightforward manner. It is given in Table 1[18] for the algebra Oc(E_6).

Once toric matrices with one twist are determined, the knowledge of this multiplication table allows one to determine all toric matrices with two twists ("second algorithm"). Besides the general property $W_{x,y} = W_{y,x}$, which holds in the present case since Oc(E_6) is commutative, this table also allows one to obtain many other identities between toric matrices; for instance, from the fact that $(1\dot{\otimes}1)\cdot(0\dot{\otimes}4) = (5\dot{\otimes}1) = (4\dot{\otimes}1)\cdot(1\dot{\otimes}0) = (5\dot{\otimes}1)\cdot(0\dot{\otimes}0)$ we deduce the identities $W_{11,04} = W_{51,00} = W_{41,10}$.

3.1.7. Toric matrices W_{xy} and frustrated functions with two twists (results)

Since $\dim(\mathrm{Oc}(E_6)) = 12$, we have a priori, 12^2 generalized toric structures $W_{x,y}$ for the graph E_6. However, taking into account the symmetry $W_{x,y} = W_{yx}$ and other identities encoded by the previous table, it happens that only 36, among the expected 144 toric structures, are distinct. It is interesting to restrict our attention to those that are symmetric, but we already know that six among the 12 toric matrices with one twist are symmetric (the three ambichiral ones $W_{00,00}$, $W_{30,00}$, $W_{40,00}$ and the three which are neither ambichiral nor chiral, $W_{11,00}$, $W_{21,00}$, $W_{51,00}$). Therefore, we are left with only six new matrices that are

[18] Part of this multiplication table was obtained by Schieber.

Table 1
Multiplication table for the Ocneanu algebra of E_6

	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{1'}$	$\underline{11'}$	$\underline{21'}$	$\underline{31'}$	$\underline{41'}$	$\underline{51'}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{1'}$	$\underline{11'}$	$\underline{21'}$	$\underline{31'}$	$\underline{41'}$	$\underline{51'}$
$\underline{1}$	$\underline{1}$	$\underline{0}+\underline{2}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{2}$	$\underline{5}$	$\underline{2}+\underline{4}$	$\underline{11'}$	$\underline{1'}+\underline{21'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{21'}$	$\underline{51'}$	$\underline{21'}+\underline{41'}$
$\underline{2}$	$\underline{2}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{0}+\underline{2}+\underline{2}+\underline{4}$	$\underline{1}+\underline{5}$	$\underline{2}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{21'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{1'}+\underline{21'}+\underline{21'}+\underline{41'}$	$\underline{11'}+\underline{51'}$	$\underline{21'}$	$\underline{11'}+\underline{31'}+\underline{51'}$
$\underline{3}$	$\underline{3}$	$\underline{2}$	$\underline{1}+\underline{5}$	$\underline{0}+\underline{4}$	$\underline{3}$	$\underline{2}$	$\underline{31'}$	$\underline{21'}$	$\underline{11'}+\underline{51'}$	$\underline{1'}+\underline{41'}$	$\underline{31'}$	$\underline{21'}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{41'}$	$\underline{51'}$	$\underline{21'}$	$\underline{31'}$	$\underline{1'}$	$\underline{11'}$
$\underline{5}$	$\underline{5}$	$\underline{2}+\underline{4}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{2}$	$\underline{1}$	$\underline{0}+\underline{2}$	$\underline{51'}$	$\underline{21'}+\underline{41'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{21'}$	$\underline{11'}$	$\underline{1'}+\underline{21'}$
$\underline{1'}$	$\underline{1'}$	$\underline{11'}$	$\underline{21'}$	$\underline{31'}$	$\underline{41'}$	$\underline{51'}$	$\underline{0}+\underline{31'}$	$\underline{1}+\underline{21'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{3}+\underline{1'}+\underline{41'}$	$\underline{4}+\underline{31'}$	$\underline{5}+\underline{21'}$
$\underline{11'}$	$\underline{11'}$	$\underline{1'}+\underline{21'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{21'}$	$\underline{51'}$	$\underline{21'}+\underline{41'}$	$\underline{1}+\underline{21'}$	$\underline{0}+\underline{2}+\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{1}+\underline{3}+\underline{5}+\underline{1'}+\underline{21'}+\underline{21'}+\underline{41'}+\underline{41'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{5}+\underline{21'}$	$\underline{2}+\underline{4}+\underline{11'}+\underline{31'}+\underline{51'}$
$\underline{21'}$	$\underline{21'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{1'}+\underline{21'}+\underline{21'}+\underline{41'}$	$\underline{11'}+\underline{51'}$	$\underline{21'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{1}+\underline{3}+\underline{5}+\underline{1'}+\underline{21'}+\underline{21'}+\underline{41'}$	$\underline{0}+\underline{2}+\underline{2}+\underline{4}+\underline{11'}+\underline{11'}+\underline{31'}+\underline{31'}+\underline{51'}+\underline{51'}$	$\underline{1}+\underline{5}+\underline{21'}+\underline{21'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{1}+\underline{3}+\underline{5}+\underline{1'}+\underline{21'}+\underline{21'}+\underline{41'}$
$\underline{31'}$	$\underline{31'}$	$\underline{21'}$	$\underline{11'}+\underline{51'}$	$\underline{1'}+\underline{41'}$	$\underline{31'}$	$\underline{21'}$	$\underline{3}+\underline{1'}+\underline{41'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{1}+\underline{5}+\underline{21'}+\underline{21'}$	$\underline{0}+\underline{4}+\underline{31'}+\underline{31'}$	$\underline{3}+\underline{1'}+\underline{41'}$	$\underline{2}+\underline{11'}+\underline{51'}$
$\underline{41'}$	$\underline{41'}$	$\underline{51'}$	$\underline{21'}$	$\underline{31'}$	$\underline{1'}$	$\underline{11'}$	$\underline{4}+\underline{31'}$	$\underline{5}+\underline{21'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{3}+\underline{1'}+\underline{41'}$	$\underline{0}+\underline{31'}$	$\underline{1}+\underline{21'}$
$\underline{51'}$	$\underline{51'}$	$\underline{21'}+\underline{41'}$	$\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{21'}$	$\underline{11'}$	$\underline{1'}+\underline{21'}$	$\underline{5}+\underline{21'}$	$\underline{2}+\underline{4}+\underline{11'}+\underline{31'}+\underline{51'}$	$\underline{1}+\underline{3}+\underline{5}+\underline{1'}+\underline{21'}+\underline{21'}+\underline{41'}$	$\underline{2}+\underline{11'}+\underline{51'}$	$\underline{1}+\underline{21'}$	$\underline{0}+\underline{2}+\underline{11'}+\underline{31'}+\underline{51'}$

given below

$$W_{30,21}=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0\\0&1&0&1&0&2&0&1&0&1&0\\0&0&2&0&2&0&2&0&2&0&0\\0&1&0&1&0&2&0&1&0&1&0\\0&0&2&0&2&0&2&0&2&0&0\\0&2&0&2&0&4&0&2&0&2&0\\0&0&2&0&2&0&2&0&2&0&0\\0&1&0&1&0&2&0&1&0&1&0\\0&0&2&0&2&0&2&0&2&0&0\\0&1&0&1&0&2&0&1&0&1&0\\0&0&0&0&0&0&0&0&0&0&0\end{pmatrix},$$

$$W_{11,51}=\begin{pmatrix}0&0&1&0&2&0&1&0&1&0&1\\0&1&0&2&0&2&0&2&0&1&0\\1&0&2&0&3&0&3&0&2&0&1\\0&2&0&4&0&4&0&4&0&2&0\\2&0&3&0&4&0&5&0&3&0&1\\0&2&0&4&0&4&0&4&0&2&0\\1&0&3&0&5&0&4&0&3&0&2\\0&2&0&4&0&4&0&4&0&2&0\\1&0&2&0&3&0&3&0&2&0&1\\0&1&0&2&0&2&0&2&0&1&0\\1&0&1&0&1&0&2&0&1&0&0\end{pmatrix},$$

$$W_{11,21}=\begin{pmatrix}0&1&0&2&0&2&0&2&0&1&0\\1&0&2&0&3&0&3&0&2&0&1\\0&2&0&4&0&4&0&4&0&2&0\\2&0&4&0&6&0&6&0&4&0&2\\0&3&0&6&0&6&0&6&0&3&0\\2&0&4&0&6&0&6&0&4&0&2\\0&3&0&6&0&6&0&6&0&3&0\\2&0&4&0&6&0&6&0&4&0&2\\0&2&0&4&0&4&0&4&0&2&0\\1&0&2&0&3&0&3&0&2&0&1\\0&1&0&2&0&2&0&2&0&1&0\end{pmatrix},$$

$$W_{03,03} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$W_{51,51} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 2 & 0 \\ 1 & 0 & 3 & 0 & 5 & 0 & 4 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 & 4 & 0 & 5 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$W_{21,21} = \begin{pmatrix} 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 & 6 & 0 & 6 & 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 4 & 0 \\ 3 & 0 & 6 & 0 & 9 & 0 & 9 & 0 & 6 & 0 & 3 \\ 0 & 4 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 4 & 0 \\ 3 & 0 & 6 & 0 & 9 & 0 & 9 & 0 & 6 & 0 & 3 \\ 0 & 4 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 4 & 0 \\ 2 & 0 & 4 & 0 & 6 & 0 & 6 & 0 & 4 & 0 & 2 \\ 0 & 2 & 0 & 4 & 0 & 4 & 0 & 4 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

Using our “first algorithm”, the last toric matrix $W_{21,21}$, for instance, is

$$W_{21,21} = \mathcal{V}_{125}[2] \cdot W_1(A_3) \cdot \mathcal{V}_{125}[1].$$

The structure of the corresponding partition function $Z_{21,21} = \bar{\chi} \cdot W_{21,21} \cdot \chi$ is better understood if we use this last expression, leading to

$$Z_{21,21} = (\bar{v}[2]_1 + \bar{v}[2]_3)v[1]_2 + \bar{v}[2]_2(v[1]_1 + v[1]_3)$$

with

$$\begin{aligned}
&v[2]_1 = \chi_2 + 2\chi_4 + 2\chi_6 + 2\chi_8 + \chi_{10}, \qquad v[1]_1 = \chi_1 + \chi_3 + \chi_5 + 2\chi_7 + \chi_9,\\
&v[2]_2 = \chi_1 + 2\chi_3 + 3\chi_5 + 3\chi_7 + 2\chi_9 + \chi_{11},\\
&v[1]_2 = \chi_2 + 2\chi_4 + 2\chi_6 + 2\chi_8 + \chi_{10}, \qquad v[2]_3 = \chi_2 + 2\chi_4 + 2\chi_6 + 2\chi_8 + \chi_{10},\\
&v[1]_3 = \chi_3 + 2\chi_5 + \chi_7 + \chi_9 + \chi_{11}
\end{aligned}$$

then if we just consider its fully developed form obtained by using the explicit expression for matrix $W_{21,21}$. The same toric matrix can also be obtained, using our “second algorithm”, as a linear combination of toric matrices with a single twist; indeed, the $\mathrm{Oc}(E_6)$ multiplication table tells us that

$$\begin{aligned}
(2\dot{\otimes}1)(2\dot{\otimes}1) &= 0\otimes 0 + 2\otimes 0 + 2\dot{\otimes}0 + 4\otimes 0 + 1\otimes 1 + 1\dot{\otimes}1 + 3\dot{\otimes}1 + 3\otimes 1\\
&\quad + 5\dot{\otimes}1 + 5\otimes 1,
\end{aligned}$$

therefore, once we have determined the toric matrices with a single twist, we get

$$W_{21,21} = W_{00,00} + 2W_{20,00} + W_{40,00} + 2W_{11,00} + 2W_{31,00} + 2W_{51,00}.$$

3.1.8. *Modular properties of E_6*

The modular matrix S for A_{11}. As discussed in Section 2.6.1, rather than using a general formula, we determine directly the modular matrix S from the properties of the diagram A_{11}. The following table gives, for each eigenvalue δ of the adjacency matrix of A_{11} (so $k = 10, \kappa = 12$), the components of the associated eigenvector ψ, chosen in such a way that it takes the value 1 at the vertex τ_0. The table also gives the norm $\bar{\psi}\psi$. Define ϕ as the normalized eigenvector corresponding to ψ (i.e., $\phi = \psi/\sqrt{\bar{\psi}\psi}$). The modular matrix S is then obtained from the table s of the 11 eigenvectors ϕ (with our conventions, $S = s$):

δ	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	$\overline{\psi\psi}$
$\beta = \sqrt{2+\sqrt{3}}$	[1]	$[2] = \beta$	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	$24(2+\sqrt{3})$
$\sqrt{3}$	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0	−1	$-\sqrt{3}$	−2	$-\sqrt{3}$	−1	24
$\sqrt{2}$	1	$\sqrt{2}$	1	0	−1	$-\sqrt{2}$	−1	0	1	$\sqrt{2}$	1	12
1	1	1	0	−1	−1	0	1	1	0	−1	−1	8
$\sqrt{2-\sqrt{3}}$	[1′]	[2′]	[3′]	[4′]	[5′]	[6′]	[7′]	[8′]	[9′]	[10′]	[11′]	$24(2-\sqrt{3})$
0	1	0	−1	0	1	0	−1	0	1	0	−1	6
$-\sqrt{2-\sqrt{3}}$	[1′]	−[2′]	[3′]	−[4′]	[5′]	−[6′]	[7′]	−[8′]	[9′]	−[10′]	[11′]	$24(2-\sqrt{3})$
−1	1	−1	0	1	−1	0	1	−1	0	1	−1	8
$-\sqrt{2}$	1	$-\sqrt{2}$	1	0	−1	$\sqrt{2}$	−1	0	1	$-\sqrt{2}$	1	12
$-\sqrt{3}$	1	$-\sqrt{3}$	2	$-\sqrt{3}$	1	0	−1	$\sqrt{3}$	−2	$\sqrt{3}$	−1	24
$-\beta = -\sqrt{2+\sqrt{3}}$	[1]	−[2]	[3]	−[4]	[5]	−[6]	[7]	−[8]	[9]	−[10]	[11]	$24(2+\sqrt{3})$

With $[n]\doteq[n]_+$, $[n']\doteq[n]_-$ and

$$[1]_\pm = [11]_\pm = 1, \qquad [2]_\pm = [10]_\pm = \sqrt{2 \pm \sqrt{3}}, \qquad [3]_\pm = [9]_\pm = 1 \pm \sqrt{3},$$
$$[4]_\pm = [8]_\pm = \pm\sqrt{3(2 \pm \sqrt{3})}, \qquad [5]_\pm = [7]_\pm = 2 \pm \sqrt{3}, \qquad [6]_\pm = 2\sqrt{2 \pm \sqrt{3}}.$$

The modular matrix T for A_{11}.

$$T = \mathrm{e}^{\mathrm{i}\pi/24}\,\mathrm{diag}[7, 10, 15, 22, -17, -6, 7, 22, -9, 10, -17].$$

Characters of A_{11}. In a neighborhood of $\mathrm{i}\infty$, the characters $\chi_r^{(10)} \doteq \xi_{r-1}^{(10)}$ of A_{11} read[19]

$$\chi_1^{(10)}(q) = q^{-5/48}(1 + 3q + 9q^2 + \cdots), \quad \chi_2^{(10)}(q) = q^{-1/24}(2 + 6q + 18q^2 + \cdots),$$
$$\chi_3^{(10)}(q) = q^{1/16}(3 + 9q + 27q^2 + \cdots), \quad \chi_4^{(10)}(q) = q^{5/24}(4 + 12q + 36q^2 + \cdots),$$
$$\chi_5^{(10)}(q) = q^{19/48}(5 + 15q + 45q^2 + \cdots), \quad \chi_6^{(10)}(q) = q^{5/8}(6 + 18q + 54q^2 + \cdots),$$
$$\chi_7^{(10)}(q) = q^{43/48}(7 + 21q + 63q^2 + \cdots),$$
$$\chi_8^{(10)}(q) = q^{29/24}(8 + 24q + 72q^2 + \cdots),$$
$$\chi_9^{(10)}(q) = q^{25/16}(9 + 27q + 81q^2 + \cdots),$$
$$\chi_{10}^{(10)}(q) = q^{47/24}(10 + 30q + 76q^2 + \cdots),$$
$$\chi_{11}^{(10)}(q) = q^{115/48}(11 + 20q + 60q^2 + \cdots).$$

Modular properties for the twisted partition functions of E_6. Since $\kappa = 12$ is even, the representation of the modular group factorizes over the finite group $\mathrm{SL}(2, \mathbb{Z}/4\kappa\mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}/48\mathbb{Z})$. A presentation of $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, by generators and relations, for $N > 4$, can be found in [14]; all necessary relations can be checked here (in particular $T^{48} = 1$). Notice that, since $48 = 16 \times 3$ and since integers 16 and 3 are relatively prime, this finite group is isomorphic with $\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}/2^4\mathbb{Z})$, of order 24×3072.

The 11-dimensional vector space spanned by the characters of A_{11} carry a representation of $\mathrm{SL}(2, \mathbb{Z}/48\mathbb{Z})$ which is not irreducible since the three-dimensional vector subspace spanned by vectors $w_1 = w[0]_1$, $w_2 = w[0]_2$ and $w_3 = w[0]_2$ is invariant. Indeed, under $S : \tau \mapsto -1/\tau$, $w_1 \mapsto (1/2)(w_1 + w_2) - (1/\sqrt{2})w_2$, $w_2 \mapsto (1/\sqrt{2})(w_3 - w_1)$, $w_3 \mapsto (1/2)(w_1 + w_2) + (1/\sqrt{2})w_2$, and under $T : \tau \mapsto \tau + 1$, $w_1 \mapsto \mathrm{e}^{19\mathrm{i}\pi/24}w_1$, $w_2 \mapsto \mathrm{e}^{5\mathrm{i}\pi/12}w_2$, $w_3 \mapsto \mathrm{e}^{-5\mathrm{i}\pi/24}w_3$. Bilinear forms on this three-dimensional irreducible subspace build a vector space $\mathbb{C}^3 \otimes \mathbb{C}^3 \simeq \mathbb{C}^9$, which itself contains an irreducible subspace of dimension 1, spanned by the $W_{0,0}$ matrix. The above transformation properties of characters allow one to check that $Z_{0,0}$ is indeed invariant, but it is much easier to check that the toric matrix $W_{0,0}$ commutes with both S and T.

Twisted partition functions are not, a priori, invariant under the modular group. By inspection, we found the following remarkable property: besides $W_{00,00}$ itself, none[20] of the

[19] Of course ... the coefficients of $\xi_j^{(k)}$ are not simply $j + 1$ times bigger than those of $\xi_0^{(k)}$!

[20] Actually $(0\dot{\otimes}4)(0\dot{\otimes}4) = 0\dot{\otimes}0$ in the multiplication table of $\mathrm{Oc}(E_6)$, so that $W_{04,04} = W_{00,00}$ and the corresponding entry in the table (it commutes with T!) is just the usual modular invariant.

toric matrices $W_{x,y}$ commutes with S, but they all commute with the operator $ST^{-1}S$; moreover, toric matrices also commute with particular powers of the operator T. The results are summarized in the following table: columns and rows are labeled by vertices x, y of the Ocneanu graph, the corresponding entry gives the smallest power p, such that $W_{x,y}$ commutes with T^p; dots stand for $p = 48$ (but this commutation property is trivial since $T^{48} = 1$ anyway):

$W_{x,y}$	00	03	04	10	20	50	01	02	05	11	21	51
00	1	16	2	.	12	.	.	12	.	12	.	12
03	16	2	16	12	.	12	12	.	12	.	12	.
04	2	16	1	.	12	.	.	12	.	12	.	12
10	.	12	.	12	.	12	12	.	12	.	12	.
20	12	.	12	.	12	.	.	12	.	12	.	12
50	.	12	.	12	.	12	12	.	12	.	12	.
01	.	12	.	12	.	12	12	.	12	.	12	.
02	12	.	12	.	12	.	.	12	.	12	.	12
05	.	12	.	12	.	12	12	.	12	.	12	.
11	12	.	12	.	12	.	.	12	.	12	.	12
21	.	12	.	12	.	12	12	.	12	.	12	.
51	12	.	12	.	12	.	.	12	.	12	.	12

The operator T^N represents the shift $\tau \mapsto \tau + N$ and $ST^{-1}S$ represents the transformation $\tau \mapsto \tau/\tau + 1$. Together, these two elements generate $\Gamma_0(N)$, a congruence subgroup of level N. The usual partition function is invariant with respect to the modular[21] group, but twisted partition functions are invariant only with respect to appropriate congruence subgroups. For instance, $Z_{03,03}$ is invariant under the subgroup $\Gamma_0(2)$. Actually, we should remember that, in this case, the whole representation factorizes through the principal congruence subgroup $\Gamma(48)$.

3.2. *Example of an affine model of type $SU(3)$: the $\mathcal{E}_5$ case*

The Di Francesco–Zuber $G = \mathcal{E}_5$ diagram is displayed in Fig. 5, it is a module over the $\mathcal{A}_5$ diagram (the generator corresponding to the given orientation is the vertex $(1, 0)$).

Induction–restriction rules between these two diagrams and determination of the corresponding Ocneanu graph was analyzed in [12]. The dimension of the space of paths on $\mathcal{E}_5$ is infinite, but when we restrict our attention to essential paths (one type of essential path for every vertex of $\mathcal{A}_5$), one finds 21 possibilities, i.e., 21 blocks of dimensions (d_j, d_j) for the first algebra structure of $\mathcal{B}G$. The integers d_j are given by the list: (12), (24, 24), (36, 48, 36), (36, 60, 60, 36), (24, 48, 60, 48, 24), (12, 24, 36, 36, 24, 12).

For its other multiplicative structure, $\mathcal{B}G$ has 24 blocks. Its dimensions d_x are as follows: six blocks with $d_x = 12$, 12 blocks with $d_x = 24$ and six blocks with $d_x = 60$.

[21] We write "modular" but the relevant group is $SL(2, \mathbb{Z})$, not $PSL(2, \mathbb{Z})$.

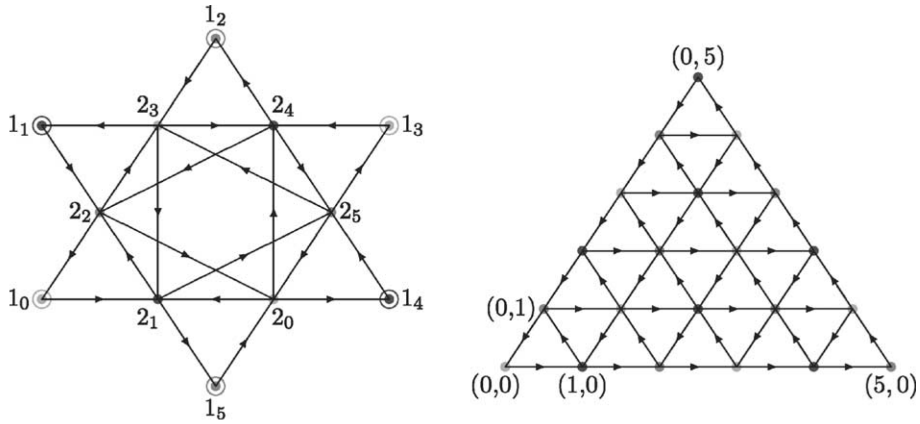


Fig. 5. The $\mathcal{E}_5$ and $\mathcal{A}_5$ generalized Dynkin diagrams.

Notice that $\sum_j d_j^2 = 29376$ and $\sum_x d_x^2 = 29376$; moreover $\sum_p d_j = 720$ and $\sum_x d_x = 720$. The indexing set for x, i.e., the Ocneanu graph of $\mathcal{E}_5$, has 24 points; it was obtained in [12] and is displayed in Fig. 8.

One obtains in this way 24 toric matrices (and partition functions) of type $W_{x,0}$, and 24^2 matrices of type $W_{x,y}$. Many of them happen to coincide. The modular invariant partition function is associated with $W_{0,0}$ and is given by

$$\begin{aligned}\mathcal{Z}_{\mathcal{E}_5} \doteq \mathcal{Z}_{1_0 \dot{\otimes} 1_0} &= |\chi_{(1,1)} + \chi_{(3,3)}|^2 + |\chi_{(1,3)} + \chi_{(4,3)}|^2 + |\chi_{(3,1)} + \chi_{(3,4)}|^2 \\ &\quad + |\chi_{(3,2)} + \chi_{(1,6)}|^2 + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + |\chi_{(2,3)} + \chi_{(6,1)}|^2.\end{aligned}$$

It agrees with the expression first obtained by Gannon [20], using entirely different techniques. One could then determine all toric matrices with one or two twists and perform the same kind of analysis as the one that was carried out for the E_6 diagram.

4. From graphs to minimal models

4.1. Central charges

Affine SU(2) models. These are the models considered in the last section; they are associated with an ADE diagram G of level k (with Coxeter number, or altitude, $\kappa = k + 2$). For an affine Lie algebra $\hat{g}_k$ at level k, the central charge is obtained from the modular phase (or from the expression of the modular T operator, see Section 2.6.3, or from the principal part of characters near complex infinity, see Section 2.7) equal to $\dim(g) \cdot k/(k + \text{Coxeter}(g))$. SU(2) models have therefore a central charge $c(k) \doteq 3k/(k + 2)$, value that we may *define* as the central charge of the underlying ADE diagram. All these models are unitary ($c \geq 1$). The limiting case $c = 1$ is obtained for $k = 1$, i.e., for the graph A_2.

Affine SU(2) models can be identified with WZW $\widehat{su}(2)_k$ models or with coset models obtained from conformal embeddings (i.e., same central charge) of type $\widehat{su}(2)_k \subset \hat{g}_1$, here

$\hat{g}_1$ is some affine Lie algebra at level 1. For instance, both models associated with diagrams A_{11} and E_6 have the same central charge ($c = 5/2$) and the second model can be obtained from a conformal embedding $\widehat{su}(2)_{10} \subset \widehat{spin}(5)_1$; we can check that $\dim(\mathrm{Spin}(5)) = 10$, $\mathrm{Coxeter}(\mathrm{Spin}(5)) = 3$, and $3.10/(10+2) = 10.1/(1+3)$.

Minimal models. Minimal models of type $\mathcal{W}_2$ (or "minimal models", for short[22]) are defined by a pair of diagrams (G_1, G_2) belonging to the SU(2) system, i.e., two ADE diagrams. We call k_1 and k_2 their respective levels (so that Coxeter numbers κ_1 and κ_2 are, respectively, equal to k_1+2 and k_2+2). Assuming that the Coxeter numbers of the diagrams are relatively prime, the general formula for the central charge is

$$c(k_1, k_2) = 1 - (k_1 - k_2)(c(k_1) - c(k_2)) = 1 - 6\frac{(\kappa_1 - \kappa_2)^2}{\kappa_1\kappa_2}.$$

Unitary minimal models are obtained when $|\kappa_1 - \kappa_2| = 1$, then $c = 1 - (6/\kappa_1\kappa_2)$ and $(0 < c < 1)$. Ordering $k_1 = k-1 < k_2 = k$, one gets $c(k-1, k) = 1-(6/(k+1)(k+2))$, which is the value obtained in particular for models of type $(\mathcal{A}_{k-1}, \mathcal{A}_k) \equiv (A_k, A_{k+1})$. The ordered set of values starts with $\{0, 1/2, 7/10, 4/5, 6/7, 25/28, \dots\}$. The limiting case $c = 0$ is obtained for the pair (A_1, A_2). In particular, $c = 21/22$ for the (A_{10}, A_{11}) model; the same value of c is obtained for the (A_{10}, E_6) model. The previous formula giving c, for unitary minimal models, can be written $c(k-1, k) = c(1)+c(k-1)-c(k)$, indeed $c(1) = 1$. This expression is therefore compatible with a coset model $\widehat{SU(2)}_{k-1} \otimes \widehat{SU(2)}_1/\widehat{SU(2)}_k$, and it is a particular case of a more general formula, valid for coset models $\hat{g}_{k_1} \otimes \hat{g}_{k_2}/\hat{g}_{k_1+k_2}$, namely $c(k_1, k_2) = \dim(g)[(k_1/(k_1+h)) + (k_2/(k_2+h)) - ((k_1+k_2)/(k_1+k_2+h))]$, where h is the dual Coxeter number of g.

Affine SU(3) models. These are the models considered in the last section and associated with a Di Francesco–Zuber diagram G of level k (with generalized Coxeter number, or altitude, $\kappa = k+3$). From the general formula for the modular phase, we see that all affine SU(3) models have a central charge $c(k) \doteq 8k/(k+3)$. All these models are unitary ($c \geq 2$). The limiting case $c = 2$ is obtained for $k = 1$, i.e., for the graph $\mathcal{A}_1$.

Affine SU(3) models can be identified with WZW $\widehat{su}(3)_k$ models or with coset models obtained from conformal embeddings (i.e., same central charge) of type $\widehat{su}(3)_k \subset \hat{g}_1$, here $\hat{g}_1$ is some affine Lie algebra at level 1. For instance, both models associated with diagrams $\mathcal{A}_5$ and $\mathcal{E}_5$ have the same central charge ($c = 5$) but the second model can be obtained from a conformal embedding $\widehat{su}(3)_5 \subset \widehat{su}(6)_1$; we can check that $\dim(\mathrm{SU}(6)) = 35$ and $\mathrm{Coxeter}(\mathrm{SU}(6)) = 6$, so that $(8 \times 5)/(5+3) = (35 \times 1)/(6+1)$.

Minimal models of type $\mathcal{W}_3$. Minimal models of type $\mathcal{W}_3$ are defined by a pair of diagrams (G_1, G_2) belonging to the SU(3) system, i.e., two Di Francesco–Zuber diagrams. We call k_1 and k_2 their respective levels, so that the generalized Coxeter numbers κ_1 and κ_2 are, respectively, equal to k_1+3 and k_2+3. Again, assuming that the Coxeter numbers of the diagrams are relatively prime, the general formula for the central charge is

$$c(k_1, k_2) = 2 - (k_1 - k_2)(c(k_1) - c(k_2)) = 2\left(1 - 12\frac{(\kappa_1 - \kappa_2)^2}{\kappa_1\kappa_2}\right).$$

[22] $\mathcal{W}_2$ denotes the Virasoro algebra.

Unitary minimal models of type $\mathcal{W}_3$ are obtained when $|\kappa_1 - \kappa_2| = 1$, then $c = 2(1 - 12/\kappa_1\kappa_2)$ and $(4/5 \le c < 2)$. Ordering $k_1 = k - 1 < k_2 = k$, one gets $c(k-1, k) = c = 2 - (24/(k+2)(k+3))$, and this holds in particular for models of type $(\mathcal{A}_{k-1}, \mathcal{A}_k)$. The ordered set of values starts with $\{4/5, 6/5, 10/7, 11/7, 5/3, \dots\}$. In particular, $c = 11/7$ for the $(\mathcal{A}_4, \mathcal{A}_5)$ model; the same value of c is obtained for the $(\mathcal{A}_4, \mathcal{E}_5)$ model. The limiting case (which is rather special), $c = 4/5 < 1$ is obtained for the pair $(\mathcal{A}_1, \mathcal{A}_2)$. For unitary models, the central charge can also be written as $c(k-1, k) = c(1) + c(k-1) - c(k)$, indeed $c(1) = 2$. This expression is therefore compatible with a coset model $\widehat{\mathrm{SU}(3)}_{k-1} \otimes \widehat{\mathrm{SU}(3)}_1 / \widehat{\mathrm{SU}(3)}_k$, and is a particular case of an already mentioned more general formula, valid for all coset models.

Remark. Minimal models of type $\mathcal{W}_N$ involve, by definition, a finite number of irreps of the algebra $\mathcal{W}_N$. The Virasoro algebra $\mathcal{W}_2$ is subalgebra of $\mathcal{W}_N$, for $N > 2$ and, in particular, of $\mathcal{W}_3$. Under the restriction ("branching rules") $\mathcal{W}_3 \mapsto \mathcal{W}_2$, an irreducible representation of $\mathcal{W}_3$ can be decomposed as a sum of irreps of $\mathcal{W}_2$, but this sum is in general infinite. For this reason, $\mathcal{W}_3$-minimal models do not give rise, in general, to usual $(\mathcal{W}_2)$ minimal models, although this may happens: it is the case for the smallest member $(\mathcal{A}_1, \mathcal{A}_2)$ of the diagonal $\mathcal{W}_3$ series (its central charge 4/5 is smaller than 1) which can be identified with the Potts model, i.e., the non-diagonal minimal model (A_4, D_4).

Affine SU(N) models and minimal models of type $\mathcal{W}_N$. Let us just mention that for a diagram of level k belonging to a generalized Coxeter–Dynkin system of type SU(N), the altitude is $\kappa = k + N$, the central charge is $c(k) = (N^2 - 1)k/(k + N)$. A minimal model of type $\mathcal{W}_N$ is defined by a pair of such diagrams and the central charge is

$$\begin{aligned} c(k_1, k_2) &= (N-1) - (k_1 - k_2)(c(k_1) - c(k_2)) \\ &= (N-1)\left(1 - N(N+1)\frac{(\kappa_1 - \kappa_2)^2}{\kappa_1\kappa_2}\right). \end{aligned}$$

More generally, if we replace SU(N) by a Lie group of rank r and dual Coxeter number N, the last formula reads [4]

$$c(k_1, k_2) = r\left(1 - N(N+1)\frac{(\kappa_1 - \kappa_2)^2}{\kappa_1\kappa_2}\right).$$

In the later case the concept of $\mathcal{W}_N$ algebras has to be generalized.

4.2. Characters, symmetry of Kac tables and partition functions

A generalized minimal model is defined by a pair (G_1, G_2) of diagrams which are members of some (generalized) Coxeter–Dynkin system. Characters are now labeled by a pair (r, s) of vertices belonging to $\mathcal{A}(G_1) \times \mathcal{A}(G_2)$, where $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ refer to the diagrams of the $\mathcal{A}$ series which have, respectively, same Coxeter number (or altitude) as the given two diagrams. As it will be recalled below, in the case of minimal models of type $\mathcal{W}_N$, what matters is a quotient of this product of diagrams by the $\mathbb{Z}_N$ group.

Minimal models. Vertices are labeled by integers r or s and we have a $\mathbb{Z}_2$ action[23] on (A_{k_1+1}, A_{k_2+1}): $(r, s) \mapsto (\sigma(r) \doteq k_1 + 2 - r, \sigma(s) \doteq k_2 + 2 - s)$. We take $1 \le r \le k_1 + 1$ and $1 \le s \le k_2 + 1$.

Minimal models of type $\mathcal{W}_3$. Vertices are labeled by SU(3) Young diagrams or by the (integer) components (r_1, r_2) of the chosen vertex with respect to the two fundamental weights of SU(3), and we have a $\mathbb{Z}_3$ action[24] on $(\mathcal{A}_{k_1}, \mathcal{A}_{k_2})$, with $(r = (r_1, r_2), s = (s_1, s_2)) \mapsto \sigma(r) \doteq ((k_1 + 3 - (r_1 + r_2), r_1), \sigma(s) \doteq (k_2 + 3 - (s_1 + s_2), s_1)$. Here we take $1 \le r_1, r_2 \le k_1 + 1$ and $1 \le s_1, s_2 \le k_2 + 1$.

The different types of frustrated partition functions. Partition functions for minimal models (twisted or not) can be thought as sesquilinear forms $Z = \bar{\phi} \cdot W \cdot \phi$ and the matrix W is obtained as a tensor product of matrices $W = W(G_1) \otimes W(G_2)$, where $W(G_1)$ and $W(G_2)$ are, respectively, toric matrices for the affine models associated with diagrams G_1 and G_2. Calling k_1 and k_2 the levels of these two diagrams, we obtain in this way—for minimal models of type Virasoro—a square matrix of dimension $((k_1+1) \times (k_2+1))^2$; for minimal models of type $\mathcal{W}_3$, it is a square matrix of dimension $((k_1+1)(k_1+2)(k_2+1)(k_2+2)/4)^2$. Naively, the elements of a vector space basis on which this W matrix acts could be labeled $\chi_r \otimes \chi_s$ in the first case, and the same thing in the second, but with $r = (r_1, r_2)$ and $s = (s_1, s_2)$. However, at this point one has to take into account the $\mathbb{Z}_2$ action (or the $\mathbb{Z}_3$ action, in the case of $\mathcal{W}_3$) that identifies basis vectors labeled by (r, s) and by $(\sigma(r), \sigma(s))$. A priori, for each pair (x_1, y_1) of vertices of the Ocneanu graph $\mathrm{Oc}(G_1)$ of the diagram G_1, we have a toric matrix $W_{x_1,y_1}(G_1)$. Same thing for the diagram G_2. The most general twisted (or frustrated) partition function, for a minimal model defined by the pair (G_1, G_2) is obtained as the $\mathbb{Z}(h)$ quotient of the sesquilinear form associated with the tensor product of toric matrices $W_{x_1,y_1}(G_1) \otimes W_{x_2,y_2}(G_2)$.

Because of the $\mathbb{Z}_N$ identification ($N = 2$ for Virasoro and $N = 3$ for $\mathcal{W}_3$), the formula for partition functions reads as

$$Z = \frac{1}{N}\bar{\chi}(W'_{\underline{x},\underline{y}} \otimes W''_{\underline{z},\underline{t}})\chi.$$

Since any of the indices x_i or y_i can be equal to $\underline{0}$, we obtain the six types of twisted partition functions announced in Section 1; they are, respectively, obtained (up to a trivial permutation of the diagrams G_1, G_2) by choosing $((x, y), (z, t))$ to be of one of the following: $((0, 0), (0, 0)$, $((x, 0), (0, 0))$, $((x, y), (0, 0))$, $((x, 0), (z, 0))$, $((x, y), (z, t))$. These six cases exhaust all possibilities for a conformal theory specified by a pair of Dynkin diagrams; of course the last case is the most general since it encompasses all the others and the usual partition function is recovered when all four indices are equal to $\underline{0}$. In principle, we should denote the most general twisted partition functions of minimal models by the symbol $W_{(x_1,y_1);(x_2,y_2)}$ and remember that x_i themselves are in general given by products of the type $a\dot{\otimes}b$. To ease the reading we often drop these indices x, y when they are equal to $\underline{0} = 0\dot{\otimes}0$ and hope that this will be clear from the context. We shall examine several examples in a later section. In the so-called "diagonal cases", one takes two diagrams of

[23] $\mathbb{Z}_2$ acts separately on the two diagrams but we take the diagonal action.

[24] $\mathbb{Z}_3$ acts separately (counter-clockwise) on the two diagrams but we take the diagonal action.

type $\mathcal{A}$; the different types of matrices (and indices) introduced before coincide: $i = a = x$, $G_a = N_i = W_x$ and the undeformed toric matrix W_0 is just the unit matrix.

Torus structures versus twisted boundary conditions. The above partition functions, also called “frustrated partition functions” are not, in general, modular invariant—but they are not arbitrary either! Another definition of the same objects together with the following interpretation was given in [37] and we repeat it here. A usual partition function on a torus function can be written as $Z = \sum_{i,j} Z_{ij}\chi_i(q)\bar{\chi}_j(q)$ with $q = \exp(2i\pi\tau)$, where the calculation is made by identifying the states at the end of the cylinder through the trace operation. Then let us imagine that we incorporate the action of an operator X attached to the non-trivial cycle of the cylinder before identifying the two ends. The operator X called *twisting operator* should commute with the Virasoro operators L_n and it is invariant under a distortion of the line to which it is attached. X is therefore attached to the homotopy class of the contour C and can be thought in general as a linear combination of operators intertwining the different copies of $\mathcal{V}_i \otimes \bar{\mathcal{V}}_j$ (Verma modules corresponding to the holomorphic and antiholomorphic sectors of the theory). In other words, the effect of X is basically to twist the boundary conditions. The partition function reads as

$$Z_X = \mathrm{tr}_H\, XT = \mathrm{e}^{-2LH} \quad \text{with } [L_n, X] = [\bar{L}_n, X] = 0.$$

An explicit expression, in the presence of two twists X and Y, was written for Z_X in [37], in terms of the matrix elements of the modular operator S but we do not use this formula in our approach since we prefer to use directly the induction–restriction rules associated with the diagrams (we do not use the Verlinde formula either since the expression of the S operator itself—and not the converse—comes from the graph algebra of the $\mathcal{A}_k$ diagrams).

4.3. Conformal weights and generalized Rocha–Cariddi formulae

If our goal is only to give expressions for the (twisted) partition functions, we do not need to use any explicit expressions for the characters $\phi_{r,s}$ of minimal models but an expression of conformal weights is needed when one wants to discuss the physical contents of a given theory. Let us recall briefly these standard results.

Characters for minimal models. Call $\kappa_1 = k_1+2$ and $\kappa_2 = k_2+2$ the Coxeter numbers of the two diagrams G_1, G_2, the conformal weights are given by the Rocha–Cariddi formula

$$h_{r,s} = \frac{(r\kappa_2 - s\kappa_1)^2 - (\kappa_2 - \kappa_1)^2}{4\kappa_1\kappa_2} \quad \text{with } 1 \le r \le \kappa_1 - 1 \text{ and } 1 \le s \le \kappa_2 - 1.$$

This expression is invariant under the $\mathbb{Z}_2$ diagonal action defined previously. In the unitary cases, $|\kappa_2-\kappa_1| = 1$ and the above expression can be simplified. For unitary minimal models and near complex infinity, the Virasoro characters read

$$\phi_{r,s} \simeq q^{-c(k_1,k_2)/24+h_{r,s}}(1 + \cdots),$$

where the expression of the central charge c for a pair of ADE diagrams was recalled before; notice that one can recover the conformal weights $h_{r,s}$ from these asymptotic expressions (without using the expression of the Kac determinant).

The general expression of characters $\phi_{r,s}$, for minimal models, is

$$\frac{q^{1/24-c(\kappa_1,\kappa_2)/24}}{\eta(q)}\left(-\sum_{u=-\infty}^{+\infty} q^{(2u\kappa_1\kappa_2+r\kappa_1+s\kappa_2)^2-(\kappa_1-\kappa_2)^2/4\kappa_1\kappa_2} + \sum_{u=-\infty}^{+\infty} q^{(2u\kappa_1\kappa_2+r\kappa_1-s\kappa_2)^2-(\kappa_1-\kappa_2)^2/4\kappa_1\kappa_2}\right).$$

Both $\phi_{r,s}$ and $h_{r,s}$ are invariant under the $\mathbb{Z}_2$ action (a symmetry of the Kac table).

Relation between affine and Virasoro characters. Call χ_1 and χ_2 the two (affine) characters associated with the graph A_2. Call $\chi_r(G_1)$ and $\chi_s(G_2)$ the affine characters of the graph G_1 and G_2 and $\phi_{r,s}(G_1, G_2)$ the Virasoro characters. In the unitary case, i.e., for consecutive graphs G_i ($\kappa_2 = \kappa_1 + 1$), we have the useful relations

$$\chi_r(G_1)\chi_1 = \sum_{s,\text{odd}} \phi_{r,s}(G_1, G_2)\chi_s(G_2), \qquad \chi_r(G_1)\chi_2 = \sum_{s,\text{even}} \phi_{r,s}(G_1, G_2)\chi_s(G_2).$$

Characters for $\mathcal{W}_3$ minimal models. In that case one has to consider two conformal weights: the first one, called $h = h^{(2)}$ is usually defined as the eigenvalue of the Virasoro generator L_0 for the highest weight vector of the representation, and the other, called $h^{(3)}$, is defined as the corresponding eigenvalue for the "W_3 generator" [4,5]. These values can also be obtained from the principal parts, near complex infinity, of the $\mathcal{W}_3$ characters. We just give the formula for h; here r and s are vectors with two indices:

$$h_{r,s} = \frac{(\kappa_2 r - \kappa_1 s)\cdot(K)\cdot(\kappa_2 r - \kappa_1 s) - 2(\kappa_2 - \kappa_1)^2}{2\kappa_1\kappa_2},$$

where

$$K = (K)_{u,v} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

is the inverse Cartan matrix of sl(3). Both characters and conformal weights are invariant under the $\mathbb{Z}_3$ action ($\mathbb{Z}_3$ symmetry of the $\mathcal{W}_3$ Kac table).

4.4. Modular operators for minimal models

We just remind the reader that the general formula giving the modular operator T, for minimal models, can, for example be obtained from the prefactor giving the asymptotic form of Virasoro characters near complex infinity and that it is [16]

$$T_{(r,s);(t,u)} = \delta_{r,t}\delta_{s,u}\,\mathrm{e}^{2\mathrm{i}\pi(h_{rs}-c/24)},$$

$$S_{(m,n);(r,s)} = 2\sqrt{\frac{2}{\kappa_1\kappa_2}}(-1)^{1+nr+ms}\sin\left[\pi\frac{\kappa_2}{\kappa_1}mr\right]\sin\left[\pi\frac{\kappa_1}{\kappa_2}sn\right].$$

5. Torus structures for unitary minimal models

5.1. Examples from the ADE series

The most general twisted partition functions are of the type

$$\tfrac{1}{2}W_{x_1,y_1}(G_1)\hat{\otimes}W_{x_2,y_2}(G_2),$$

where the $\hat{\otimes}$ symbol means that we first calculate the tensor product of the toric matrices, and then identify pairwise basis elements $\phi_{r,s}$ according to the $\mathbb{Z}_2$ symmetry of the Kac table. As we know, $W_{x_1,y_1}(G_1)$ can be gotten from the knowledge of the toric matrices with only one twist $W_{x_1,0}(G_1)$ and from the multiplication table of $\mathrm{Oc}(G_1)$ which, for members of the $\mathcal{A}$ series, coincide with the graph matrices (fusion matrices) themselves, their determination is then relatively easy, see Section 2.5. We have an analogous comment for the graph G_2. For minimal models, it is therefore enough to study twisted partition functions of the type $Z_{(x_1,0);(x_2,0)} = (1/2)W_{x_1,0}(G_1)\hat{\otimes}W_{x_2,0}(G_2)$, that we shall call "fundamental toric structures", for short. Now, if we consider *unitary* cases and further restrict our attention to the so-called "diagonal cases", i.e., when both graphs are of type $\mathcal{A}$, unitarity requirement tells us that $G_1 = A_n$ and $G_2 = A_{n+1}$. In such cases, we would expect $n(n+1)$ fundamental toric structures, but $\mathbb{Z}_2$ symmetry brings this number down to $n(n+1)/2$. More general unitary models are obtained when the corresponding Coxeter numbers are consecutive integers; for pairs of diagrams involving members from the D or E series, a general determination of all fundamental toric structures has to take fully into account the structure of the corresponding Ocneanu graphs. For illustration, we shall examine three unitary cases in this section: the Ising model—it is associated with (A_2, A_3), the Potts model—it is associated with (A_4, D_4), and the (A_{10}, E_6) model.

5.1.1. Ising model

The first non-trivial case of the minimal models series corresponds to the Ising model with central charge $c = 1/2$. This model is associated with a pair of Dynkin diagrams (A_2, A_3) with Coxeter numbers $(\kappa_{A_2} = 3, \kappa_{A_3} = 4)$. The modular invariant partition function can be build from the tensor product of the corresponding fundamental toric matrices $W_0^{(A_2)} \otimes W_0^{(A_3)}$ which, respectively, describe the undeformed torus structures of the two diagrams. In what follows we present the partition functions associated with the $3 = 2\times 3/2$ fundamental toric structures, as discussed above, they are of the form $W_{x_1}^{(A_2)}\hat{\otimes}W_{x_2}^{(A_3)} \doteq W_{x_1,0}^{(A_2)}\hat{\otimes}W_{x_2,0}^{(A_3)}$. Following Petkova and Zuber [36], they can be interpreted as a result of the insertion of twisted boundary conditions (defect lines). The toric matrices of the type W_x of the pair (A_2, A_3) are

$$W_0(A_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W_1(A_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad W_0(A_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$W_1(A_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad W_2(A_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The fundamental twisted partition functions are given by

$$Z_{xy} = \tfrac{1}{2}\bar{\phi}(W_x(A_2)\hat{\otimes}W_y(A_3))\phi,$$

where $\phi = (\phi_{11}, \phi_{21}, \phi_{12}, \phi_{22}, \phi_{13}, \phi_{23})$ are the characters identifying highest weight representations with conformal dimensions given in the following table:

$h_{11} = h_{23} = 0$
$h_{12} = h_{22} = \frac{1}{16}$
$h_{13} = h_{21} = \frac{1}{2}$

Here and below, the Virasoro characters ϕ are labeled with indices r, s starting from 1 (because this is standard), but indices i, j labeling partition functions or toric matrices in the A cases, start from 0 (because this is our convention for labeling vertices of A_n diagrams). Such a choice is admittedly confusing but we hope that the reader, being warned, will not be mistaken. The characters will also be sometimes labeled by the corresponding conformal weights: $\phi_{h(r,s)} \equiv \phi_{r,s}$.

The six possible cases are listed and explicitly build as follows:

$$Z_{00} = \frac{1}{2}\bar{\phi}(W_0(A_2)\otimes W_0(A_3))\phi = \frac{1}{2}\bar{\phi}\left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & 1 & & & & \\ \hline & & 1 & & & \\ & & & 1 & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right)\phi$$

$$= \frac{1}{2}[(|\phi_{1,1}|^2 + |\phi_{2,3}|^2) + (|\phi_{1,2}|^2 + |\phi_{2,2}|^2) + (|\phi_{1,3}|^2 + |\phi_{2,1}|^2)]$$

$$= |\phi_{1,1}|^2 + |\phi_{1,2}|^2 + |\phi_{1,3}|^2 = |\phi_0|^2 + |\phi_{1/2}|^2 + |\phi_{1/16}|^2.$$

Actually, we should have written $\hat{\otimes}$ rather than $\otimes$ in the above first line, but lines 2 and 3 also involve an (hidden) $\mathbb{Z}_2$ identification, which is explicitly performed on line 4 ($\phi_{1,1} = \phi_{2,3}$, etc.). From now on we shall not mention it explicitly but it should always be understood that an $\mathbb{Z}_2$ identification of characters, corresponding to the symmetry of the Kac table (conformal weights), should be performed at the end of calculations:

$$Z_{01} = \frac{1}{2}\bar{\phi}(W_0(A_2)\otimes W_1(A_3))\phi = \frac{1}{2}\bar{\phi}\left(\begin{array}{cc|cc|cc} & & 1 & & & \\ & & & 1 & & \\ \hline 1 & & & & 1 & \\ & 1 & & & & 1 \\ \hline & & 1 & & & \\ & & & 1 & & \end{array}\right)\phi$$

$$= \phi_{1/16}(\bar{\phi}_{1/2} + \bar{\phi}_0) + \bar{\phi}_{1/16}(\phi_{1/2} + \phi_0),$$

$$Z_{02} = \frac{1}{2}\bar{\phi}(W_0(A_2) \otimes W_2(A_3))\phi = \frac{1}{2}\bar{\phi}\left(\begin{array}{cc|cc|cc} & & & & 1 & \\ & & & & & 1 \\ \hline & & 1 & & & \\ & & & 1 & & \\ \hline 1 & & & & & \\ & 1 & & & & \end{array}\right)\phi$$

$$= \bar{\phi}_0\phi_{1/2} + \bar{\phi}_{1/2}\phi_0 + |\phi_{1/16}|^2,$$

$$Z_{10} = \frac{1}{2}\bar{\phi}(W_1(A_2) \otimes W_0(A_3))\phi = \frac{1}{2}\bar{\phi}\left(\begin{array}{cc|cc|cc} & 1 & & & & \\ 1 & & & & & \\ \hline & & & 1 & & \\ & & 1 & & & \\ \hline & & & & & 1 \\ & & & & 1 & \end{array}\right)\phi$$

$$= \bar{\phi}_0\phi_{1/2} + \bar{\phi}_{1/2}\phi_0 + |\phi_{1/16}|^2 = Z_{02},$$

$$Z_{11} = \frac{1}{2}\bar{\phi}(W_1(A_2) \otimes W_1(A_3))\phi = \frac{1}{2}\bar{\phi}\left(\begin{array}{cc|cc|cc} & & & 1 & & \\ & & 1 & & & \\ \hline & 1 & & & & 1 \\ 1 & & & & 1 & \\ \hline & & & 1 & & \\ & & 1 & & & \end{array}\right)\phi$$

$$= \phi_{1/16}(\bar{\phi}_{1/2} + \bar{\phi}_0) + \bar{\phi}_{1/16}(\phi_{1/2} + \phi_0) = Z_{01},$$

$$Z_{12} = \frac{1}{2}\bar{\phi}(W_1(A_2) \otimes W_2(A_3))\phi = \frac{1}{2}\bar{\phi}\left(\begin{array}{cc|cc|cc} & & & & & 1 \\ & & & & 1 & \\ \hline & & & 1 & & \\ & & 1 & & & \\ \hline & 1 & & & & \\ 1 & & & & & \end{array}\right)\phi$$

$$= |\phi_0|^2 + |\phi_{1/2}|^2 + |\phi_{1/16}|^2 = Z_{00}.$$

We therefore obtain only three distinct partition functions, as expected: equalities $Z_{i,j} = Z_{2-i,3-j}$ are consequences of the $\phi_{r,s} = \phi_{3-r,4-s}$ identifications (remember that i, j indices are shifted by one unit, compared with r, s indices). In the graph algebra A_3, we have $\sigma_1^2 = \sigma_0 + \sigma_2$, and since $\mathrm{Oc}(A_3) = A_3$, a corresponding toric structure, with two twists, described by the matrix $W_{1;1}(A_3) = W_0(A_3) + W_2(A3)$; in the Ising model, we can therefore also consider the (non-fundamental) partition function associated with the toric matrix $W_0(A_2) \otimes W_{1;1}(A_3)$, i.e., $Z_{00} + Z_{02} = |\phi_0 + \phi_{1/2}|^2 + 2|\phi_{1/16}|^2$ [35]. We summarize the results for the fundamental partition functions in the following table:

$Z_{00} = Z_{12} = \vert\phi_0\vert^2 + \vert\phi_{1/2}\vert^2 + \vert\phi_{1/16}\vert^2$
$Z_{01} = Z_{11} = \phi_{1/16}(\bar{\phi}_{1/2} + \bar{\phi}_0) + \bar{\phi}_{1/16}(\phi_{1/2} + \phi_0)$
$Z_{02} = Z_{10} = \bar{\phi}_0\phi_{1/2} + \bar{\phi}_{1/2}\phi_0 + \vert\phi_{1/16}\vert^2$

The representations of the modular group appearing in these theories are usually not faithful: it can be checked that $T^{24} = 1$ for the affine A_2 case, $T^{16} = 1$ for the affine A_3 case and $T^{48} = 1$ for the minimal model (A_2, A_3). Z_{00}, determined above, is the usual modular invariant partition function of the Ising model, and the associated toric matrix commutes in particular with T. Toric matrices associated with Z_{02} and Z_{01}, respectively, commute with T^2 and T^{16}. The twisted partition function Z_{02} is invariant under the congruence subgroup $\Gamma_0(2)$, which involves an additional Z_2 symmetry (in general, $\Gamma_0(k)$ is not an invariant subgroup of the modular group Γ but it contains, as well as all its conjugates, the principal congruence subgroup $\Gamma(k)$, which is invariant in Γ, moreover, $\Gamma_0(k)/\Gamma(k) \simeq \mathbb{Z}_k$). In the language of twisted boundary conditions, one assumes that the fields corresponding to given characters are invariant only up to a phase under translations of the lattice, i.e., one assumes that they transform according to one-dimensional representations of the cyclic group $\mathbb{Z}_k$. In the case of $Z_{0,2}$, these are periodic boundary conditions imposed on the spin in one direction, and antiperiodic ones in the other [35]. The interpretation of Z_{01}, for which the partition function is invariant under the congruence subgroup $\Gamma_0(16)$ would be interesting to study further.

5.1.2. Potts model

As a second example we consider the (non-diagonal) Potts model with central charge $c = 4/5$. This example differs from the previous one since the pair of Dynkin diagrams involved are not both of the A_n type but (A_4, D_4) with dual Coxeter numbers $(\kappa_{A_4} = 5, \kappa_{D_4} = 6)$. Following the same steps as before we present the fundamental partition functions.

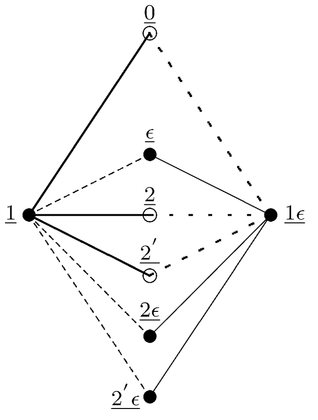
The $\mathcal{A}$ diagram corresponding to D_4 is A_5 (same Coxeter number) so that the twisted partition functions in this case take the form

$$Z_{xy} = \tfrac{1}{2}\bar{\phi}(W_x(A_4) \otimes W_y(D_4))\phi,$$

$$\phi = (\phi_{11}, \phi_{21}, \phi_{31}, \phi_{41}, \phi_{12}, \phi_{22}, \phi_{32}, \phi_{42}, \phi_{13}, \phi_{23}, \phi_{33}, \phi_{43}, \phi_{14}, \phi_{24}, \phi_{34}, \phi_{44}, \phi_{15}, \phi_{25}, \phi_{35}, \phi_{45}).$$

The table of conformal weights for the (A_4, A_5) system is given by the following table which contains $4 \times 5 = 20$ entries but only $10 = 20/2$ distinct weights. Only those weights $h_{r,s}$ such that s belongs to the set of exponents of D_4 are conformal weights for the (undeformed) (A_4, D_4) model. Exponents of D_4 are 1, 3, 5 and 3, as it is well known (or calculate the adjacency matrix of the diagram and use Section 2.3). The conformal weights obeying this criteria are typed in bold in the following table; there are only six of them. In the Virasoro minimal models language, these states (called "Potts model states") correspond only to a subset of conformal primary fields and are closed under fusion rules. Introduction of twists in general involves states of the (A_4, A_5) system which are different from the usual Potts states; the usual identification, stemming from the Z_2 symmetry, of course still holds:

$\mathbf{h}_{11} = \mathbf{h}_{45} = \mathbf{0}$
$h_{12} = h_{44} = \frac{1}{8}$
$\mathbf{h}_{13} = \mathbf{h}_{43} = \mathbf{\frac{2}{3}}$
$h_{14} = h_{42} = \frac{13}{8}$
$\mathbf{h}_{15} = \mathbf{h}_{41} = \mathbf{3}$



$$W_{00} = \begin{pmatrix} 1 & . & . & . & 1 \\ . & . & . & . & . \\ . & . & 2 & . & . \\ . & . & . & . & . \\ 1 & . & . & . & 1 \end{pmatrix}$$

Fig. 6. The D_4 Ocneanu graph and the modular invariant matrix.

$\mathbf{h}_{21} = \mathbf{h}_{35} = \mathbf{\frac{2}{5}}$
$h_{22} = h_{34} = \frac{1}{40}$
$\mathbf{h}_{23} = \mathbf{h}_{33} = \mathbf{\frac{1}{15}}$
$h_{24} = h_{32} = \frac{21}{40}$
$\mathbf{h}_{25} = \mathbf{h}_{31} = \mathbf{\frac{7}{5}}$

In order to build the fundamental toric matrices for this model, we need to use those corresponding to the Ocneanu graphs associated with diagrams A_4 and D_4. The first is easy: $\mathrm{Oc}(A_4) = A_4$. The corresponding toric matrices are

$$W_0(A_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W_1(A_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$W_2(A_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad W_3(A_4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The graph $\mathrm{Oc}(D4)$, given in [28] has eight points; it looks like two “mixed” copies of the diagram D_4 (see Fig. 6). A study of the corresponding algebra—which is non-commutative[25]—was performed in one of the sections of Coquereaux and Schieber [12]. These points are labeled 0, 1, 2, 2′ and ϵ, 1ϵ, 2ϵ, $2'\epsilon$.

There are eight generators but it turns out (see also [12]) that there are only five distinct toric matrices with one twist W_x with $x \in \{0, 2, \epsilon, 1, 1\epsilon\}$ (indeed, $W_\epsilon = W_{2\epsilon} = W_{2'\epsilon}$ and $W_2 = W_{2'}$). We call $\bar{1} \doteq 1\epsilon$. The non-commutativity of $\mathrm{Oc}(D_4)$ does not show up in

[25] Classical symmetries of the D_4 diagram are described by the non-commutative group algebra of the permutation group S_3, this non-commutativity also shows up at the quantum level in the structure of $\mathrm{Oc}(D4)$.

the fundamental twisted partition functions, indeed, although $2\epsilon \neq \epsilon 2$ in this algebra [11] (actually $\epsilon 2 = 2'\epsilon$), the toric matrices associated with these two points are the same:

$$W_0(D_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W_2(D_4) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$W_\epsilon(D_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad W_1(D_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{\bar{1}}(D_4) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

We expect $\mathbb{Z}_2$ to act as usual on A_4 but trivially on $\mathrm{Oc}(D_4)$; identification is a priori $Z_{(x_1,0);(x_2,0)} = Z_{(3-x_1,0);(x_2,0)}$, i.e., $Z_{0;x} = Z_{3;x}$ and $Z_{1;x} = Z_{2;x}$; this can be checked explicitly. So the number $4 \times 8 = 32$ of partition functions of this type reduces to 4×5 because of the accidental degeneracy between the toric matrices of D_4 (only five cases) and actually to $2 \times 5 = 10$ because of the $\mathbb{Z}_2$ identifications. In the following we list these partition functions: we have 20 fundamental toric matrices, but 10 distinct partition functions (and only four among them involving the usual Potts' model states):

$$Z_{00} = \frac{1}{2}\bar{\phi}(W_0(A_4) \otimes W_0(D_4))\phi$$

$$= \frac{1}{2}\bar{\phi}\left(\begin{array}{cccc|cccc|cccc|cccc|cccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
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\hline
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)\phi$$

$$= \frac{1}{2}\left[\sum_{r=1}^{4}(2|\phi_{r,3}|^2 + |\phi_{r,1} + \phi_{r,5}|^2)\right] = \sum_{r=1}^{2}(2|\phi_{r,3}|^2 + |\phi_{r,1} + \phi_{r,5}|^2).$$

$$
\begin{aligned}
Z_{02} &= \frac{1}{2}\bar{\phi}(W_0(A_4) \otimes W_2(D_4))\phi \\
&= \frac{1}{2}\bar{\phi}\left(\begin{array}{cccc|cccc|cccc|cccc|cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)\phi \\
&= \frac{1}{2}\left\{\sum_{r=1}^{4}[|\phi_{r,3}|^2 + (\bar{\phi}_{r,3}(\phi_{r,1} + \phi_{r,5}) + hc)]\right\} \\
&= \sum_{r=1}^{2}[|\phi_{r,3}|^2 + (\bar{\phi}_{r,3}(\phi_{r,1} + \phi_{r,5}) + hc)].
\end{aligned}
$$

From now on, we no longer display the tensor product of matrices:

$$
Z_{0\epsilon} = \frac{1}{2}\bar{\phi}(W_0(A_4) \otimes W_\epsilon(D_4))\phi = \frac{1}{2}\left[\sum_{r=1}^{4}|\phi_{r,2} + \phi_{r,4}|^2\right] = \sum_{r=1}^{2}|\phi_{r,2} + \phi_{r,4}|^2,
$$

$$
\begin{aligned}
Z_{01} &= \frac{1}{2}\bar{\phi}(W_0(A_4) \otimes W_1(D_4))\phi = \frac{1}{2}\left[\sum_{r=1}^{4}(\bar{\phi}_{r,2} + \bar{\phi}_{r,4})(\phi_{r,1} + \phi_{r,5} + 2\phi_{r,3})\right] \\
&= \sum_{r=1}^{2}[(\bar{\phi}_{r,2} + \bar{\phi}_{r,4})(\phi_{r,1} + \phi_{r,5} + 2\phi_{r,3})],
\end{aligned}
$$

$$
\begin{aligned}
Z_{0\bar{1}} &= \frac{1}{2}\bar{\phi}(W_0(A_4) \otimes W_{\bar{1}}(D_4))\phi = \frac{1}{2}\left[\sum_{r=1}^{4}(\phi_{r,2} + \phi_{r,4})(\bar{\phi}_{r,1} + \bar{\phi}_{r,5} + 2\bar{\phi}_{r,3})\right] \\
&= \sum_{r=1}^{2}[(\phi_{r,2} + \phi_{r,4})(\bar{\phi}_{r,1} + \bar{\phi}_{r,5} + 2\bar{\phi}_{r,3})] = \bar{Z}_{03},
\end{aligned}
$$

$$
\begin{aligned}
Z_{10} &= \tfrac{1}{2}\bar{\phi}(W_1(A_4) \otimes W_0(D_4))\phi \\
&= 2|\phi_{2,3}|^2 + |\phi_{2,1} + \phi_{2,5}|^2 + [(\bar{\phi}_{2,1} + \bar{\phi}_{2,5})(\phi_{1,1} + \phi_{1,5}) + 2\bar{\phi}_{2,3}\phi_{1,3} + hc],
\end{aligned}
$$

$$\begin{aligned}
Z_{12} &= \frac{1}{2}\bar{\phi}(W_1(A_4)\otimes W_2(D_4))\phi \\
&= \left[\phi_{1,3}(\bar{\phi}_{2,1}+\bar{\phi}_{2,5}+\bar{\phi}_{2,3}) + \sum_{r=1}^{2}\phi_{2,3}(\bar{\phi}_{r,1}+\bar{\phi}_{r,5}) + hc\right] + |\phi_{2,3}|^2,
\end{aligned}$$

$$\begin{aligned}
Z_{1\epsilon} &= \tfrac{1}{2}\bar{\phi}(W_1(A_4)\otimes W_\epsilon(D_4))\phi \\
&= |\phi_{2,2}+\phi_{2,4}|^2 + [(\phi_{2,2}+\phi_{2,4})(\bar{\phi}_{1,2}+\bar{\phi}_{1,4}) + hc],
\end{aligned}$$

$$\begin{aligned}
Z_{11} &= \frac{1}{2}\bar{\phi}(W_1(A_4)\otimes W_1(D_4))\phi \\
&= \sum_{r=1}^{2}[(\bar{\phi}_{r,2}+\bar{\phi}_{r,4})(\phi_{2,1}+\phi_{2,5}+2\phi_{2,3})] + (\bar{\phi}_{2,2}+\bar{\phi}_{2,4})(\phi_{1,1}+\phi_{1,5}+2\phi_{1,3}),
\end{aligned}$$

$$Z_{1\bar{1}} = \tfrac{1}{2}\bar{\phi}(W_1(A_4)\otimes W_{\bar{1}}(D_4))\phi = \bar{Z}_{13},$$

$$\begin{aligned}
Z_{20} &= \tfrac{1}{2}\bar{\phi}(W_2(A_4)\otimes W_0(D_4))\phi = Z_{10}, & Z_{22} &= \tfrac{1}{2}\bar{\phi}(W_2(A_4)\otimes W_2(D_4))\phi = Z_{12},\\
Z_{2\epsilon} &= \tfrac{1}{2}\bar{\phi}(W_2(A_4)\otimes W_\epsilon(D_4))\phi = Z_{1\epsilon}, & Z_{21} &= \tfrac{1}{2}\bar{\phi}(W_2(A_4)\otimes W_1(D_4))\phi = Z_{11},\\
Z_{2\bar{1}} &= \tfrac{1}{2}\bar{\phi}(W_2(A_4)\otimes W_{\bar{1}}(D_4))\phi = Z_{1\bar{1}}, & Z_{30} &= \tfrac{1}{2}\bar{\phi}(W_3(A_4)\otimes W_0(D_4))\phi = Z_{00},\\
Z_{32} &= \tfrac{1}{2}\bar{\phi}(W_3(A_4)\otimes W_2(D_4))\phi = Z_{02}, & Z_{3\epsilon} &= \tfrac{1}{2}\bar{\phi}(W_3(A_4)\otimes W_\epsilon(D_4))\phi = Z_{0\epsilon},\\
Z_{31} &= \tfrac{1}{2}\bar{\phi}(W_3(A_4)\otimes W_1(D_4))\phi = Z_{01}, & Z_{3\bar{1}} &= \tfrac{1}{2}\bar{\phi}(W_3(A_4)\otimes W_{\bar{1}}(D_4))\phi = Z_{0\bar{1}}.
\end{aligned}$$

The results are summarized in the following table:

$\mathbf{Z}_{00} = \mathbf{Z}_{30} = \sum_{r=1}^{2}(2\lvert\phi_{r,3}\rvert^2 + \lvert\phi_{r,1}+\phi_{r,5}\rvert^2)$
$\mathbf{Z}_{02} = \mathbf{Z}_{32} = \sum_{r=1}^{2}[\lvert\phi_{r,3}\rvert^2 + (\bar{\phi}_{r,3}(\phi_{r,1}+\phi_{r,5}) + hc)]$
$Z_{0\epsilon} = Z_{3\epsilon} = \sum_{r=1}^{2}\lvert\phi_{r,2}+\phi_{r,4}\rvert^2$
$Z_{01} = Z_{31} = \sum_{r=1}^{2}[(\bar{\phi}_{r,2}+\bar{\phi}_{r,4})(\phi_{r,1}+\phi_{r,5}+2\phi_{r,3})]$
$Z_{0\bar{1}} = Z_{3\bar{1}} = \bar{Z}_{01} = \bar{Z}_{31}$
$\mathbf{Z}_{10} = \mathbf{Z}_{20} = \mathbf{2}\lvert\phi_{2,3}\rvert^2 + \lvert\phi_{2,1}+\phi_{2,5}\rvert^2 + [(\bar{\phi}_{2,1}+\bar{\phi}_{2,5})(\phi_{1,1}+\phi_{1,5}) + 2\bar{\phi}_{2,3}\phi_{1,3} + hc]$
$\mathbf{Z}_{12} = \mathbf{Z}_{22} = \left[\phi_{1,3}(\bar{\phi}_{2,1}+\bar{\phi}_{2,5}+\bar{\phi}_{2,3}) + \sum_{r=1}^{2}\phi_{2,3}(\bar{\phi}_{r,1}+\bar{\phi}_{r,5}) + hc\right] + \lvert\phi_{2,3}\rvert^2$
$Z_{1\epsilon} = Z_{2\epsilon} = \lvert\phi_{2,2}+\phi_{2,4}\rvert^2 + [(\phi_{2,2}+\phi_{2,4})(\bar{\phi}_{1,2}+\bar{\phi}_{1,4}) + hc]$
$Z_{11} = Z_{21} = \sum_{r=1}^{2}[(\bar{\phi}_{r,2}+\bar{\phi}_{r,4})(\phi_{2,1}+\phi_{2,5}+2\phi_{2,3})] + (\bar{\phi}_{2,2}+\bar{\phi}_{2,4})(\phi_{1,1}+\phi_{1,5}+2\phi_{1,3})$
$Z_{1\bar{1}} = Z_{2\bar{1}} = \bar{Z}_{11} = \bar{Z}_{21}$

Only the partition functions written in bold letters involve the subset of states corresponding to the undeformed three states Potts model; we rewrite them by using conformal weights as field subscripts. We set $\mathbf{Z}_0 = Z_{00}$, $\mathbf{Z}_1 = Z_{02}$, $\mathbf{Z}_2 = Z_{10}$, $\mathbf{Z}_3 = Z_{12}$:

$$\mathbf{Z}_0 = 2(|\phi_{2/3}|^2 + |\phi_{1/15}|^2) + |\phi_0 + \phi_3|^2 + |\phi_{2/5} + \phi_{7/5}|^2$$

$$\mathbf{Z}_1 = |\phi_{2/3}|^2 + |\phi_{1/15}|^2 + (\bar{\phi}_{2/3}(\phi_0 + \phi_3) + \bar{\phi}_{1/15}(\phi_{2/5} + \phi_{7/5}) + hc)$$

$$\mathbf{Z}_2 = \mathbf{2}|\phi_{1/15}|^2 + |\phi_{2/5} + \phi_{7/5}|^2 + [(\bar{\phi}_{2/5} + \bar{\phi}_{7/5})(\phi_0 + \phi_3) + 2\bar{\phi}_{1/15}\phi_{2/3} + hc]$$

$$\mathbf{Z}_3 = [\phi_{2/3}(\bar{\phi}_{2/3} + \bar{\phi}_{7/5} + \bar{\phi}_{1/15}) + \phi_{1/15}(\bar{\phi}_0 + \bar{\phi}_3) + \phi_{1/15}(\bar{\phi}_{2/3} + \bar{\phi}_{7/5}) + hc] + |\phi_{1/15}|^2$$

When no twisted boundary conditions are imposed, we recover the usual modular invariant partition function $\mathbf{Z}_0$.

5.1.3. The A_{10}–E_6 example

Finally we consider the model (A_{10}–E_6) with dual Coxeter numbers ($\kappa_{A_{10}} = 11, \kappa_{E_6} = 12$) and central charge $c = 21/22$. First notice that $\mathcal{A}(E_6) = A_{11}$ so that conformal weights $h_{r,s}$ of this model (which is unitary since $11 = 10 + 1$) is a subset of the set of conformal weights for (A_{10}, A_{11}). Index r stands for A_{10} and s for A_{11}. A priori there are $10 \times 11 = 110$ possibilities, but because of the Z_2 symmetry of the Kac table ($h_{r,s} = h_{11-r,12-s}$; $1 \le r \le 10$; $1 \le s \le 11$), there are only half of them, so 55. The following table lists the conformal dimensions associated to the primary fields of this model. Only those columns $h_{r,s}$ such that s belongs to the set of exponents of E_6 are conformal weights for the usual (undeformed) (A_{10}, E_6) model. The exponents of diagram E_6 are 1, 4, 5, 7, 8 and 11 (cf. Section 2.3). Columns 1, 4, 5, 7, 8 and 11 of the following table are in boldface.

r	h_{rs}										
	$s=\mathbf{1}$	$s=2$	$s=3$	$s=\mathbf{4}$	$s=\mathbf{5}$	$s=6$	$s=\mathbf{7}$	$s=\mathbf{8}$	$s=9$	$s=10$	$s=\mathbf{11}$
1	$\mathbf{0}$	$\frac{3}{16}$	$\frac{5}{6}$	$\mathbf{\frac{31}{16}}$	$\mathbf{\frac{7}{2}}$	$\frac{265}{48}$	$\mathbf{8}$	$\mathbf{\frac{175}{16}}$	$\frac{43}{3}$	$\frac{291}{16}$	$\mathbf{\frac{45}{2}}$
2	$\mathbf{\frac{7}{22}}$	$\frac{1}{176}$	$\frac{5}{33}$	$\mathbf{\frac{133}{176}}$	$\mathbf{\frac{20}{11}}$	$\frac{1763}{528}$	$\mathbf{\frac{117}{22}}$	$\mathbf{\frac{1365}{176}}$	$\frac{703}{66}$	$\frac{2465}{176}$	$\mathbf{\frac{196}{11}}$
3	$\mathbf{\frac{13}{11}}$	$\frac{65}{176}$	$\frac{1}{66}$	$\mathbf{\frac{21}{176}}$	$\mathbf{\frac{15}{22}}$	$\frac{899}{528}$	$\mathbf{\frac{35}{11}}$	$\mathbf{\frac{901}{176}}$	$\frac{248}{66}$	$\frac{1825}{176}$	$\mathbf{\frac{301}{22}}$
4	$\mathbf{\frac{57}{22}}$	$\frac{225}{176}$	$\frac{14}{33}$	$\mathbf{\frac{5}{176}}$	$\mathbf{\frac{1}{11}}$	$\frac{323}{528}$	$\mathbf{\frac{35}{22}}$	$\mathbf{\frac{533}{176}}$	$\frac{325}{66}$	$\frac{1281}{176}$	$\mathbf{\frac{111}{11}}$
5	$\mathbf{\frac{50}{11}}$	$\frac{481}{176}$	$\frac{91}{66}$	$\mathbf{\frac{85}{176}}$	$\mathbf{\frac{1}{22}}$	$\frac{35}{528}$	$\mathbf{\frac{6}{11}}$	$\mathbf{\frac{261}{176}}$	$\frac{95}{33}$	$\frac{833}{176}$	$\mathbf{\frac{155}{22}}$

In order to build the fundamental toric matrices for this model we need to use the toric matrices (with one twist) associated with diagrams A_{10} and E_6. The more general toric matrices of the model can be got from the multiplication table of $\mathrm{Oc}(A_{10})$, which coincides with A_{10} itself (so it is easy) and the multiplication table of $\mathrm{Oc}(E_6)$ was explicitly studied in the previous section. We have 12 toric matrices $W_i(E_6)$ which are of size 11×11 and 10 toric matrices $W_i(A_{10})$ which are of size 10×10. Partition functions are then associated with matrices $Z_{(x_1,0);(x_2,0)} = (1/2)W_{x_1,0}(A_{10}) \otimes W_{x_2,0}(E_6)$ (of size 110×110). A priori, we have $10 \times 12 = 120$ of them, but because of Z_2 identifications only half of them, so 60, will be distinct. It is therefore enough to consider partition functions of the type $Z_{(x_1,0);(x_2,0)}$ for $x_1 = 0,1,2,3,4 \in A_{10}$ and x_2, a point of $\mathrm{Oc}(E_6)$, so x_2 is a member of the list: $\{(0\dot{\otimes}0, 0\dot{\otimes}3, 0\dot{\otimes}4), (1\dot{\otimes}0, 2\dot{\otimes}0, 5\dot{\otimes}0), (0\dot{\otimes}1, 0\dot{\otimes}2, 0\dot{\otimes}5), (1\dot{\otimes}1, 2\dot{\otimes}1, 5\dot{\otimes}1)\}$. Since the 12 points x_2 of $\mathrm{Oc}(E_6)$ belong to four distinct subsets of three points each (ambichiral, left chiral, right chiral, or complementary), it is natural to decompose our 60 candidates into four subsets of 15 each, labeled in the same way. The corresponding table of results is quite long ... so we shall only give the 15 twisted partition functions that belong to the first family: they are of the type $Z_{(x_1,0);(x_2,0)}$ for $x_1 = 0,1,2,3,4$ and $x_2 = \{0\dot{\otimes}0, 0\dot{\otimes}3, 0\dot{\otimes}4\}$. These partition functions are the ones that involve only the combination of characters $(1,7), (4,8), (5,11)$, i.e., the adapted vector w of Section 3.1.5. The symmetry relations for this family will read as: $Z_{(x_1,0);(0\dot{\otimes}0,0)} = Z_{(9-x_1,0);(0\dot{\otimes}4,0)}$ and $Z_{(x_1,0);(0\dot{\otimes}3,0)} = Z_{(9-x_1,0);(0\dot{\otimes}3,0)}$. Similar expressions are obtained for the other three families. Here are the explicit results.

Calling ϕ the characters vector of 110 components (only 55 distinct conformal weight), the 15 twisted partition functions involving the characters of type $\phi_{i1}+\phi_{i7}$, $\phi_{i4}+\phi_{i8}$, $\phi_{i5}+\phi_{i11}$ take the following general form:

$$Z_{ij} = \tfrac{1}{2}\bar{\phi}(W_i(A_{10}) \otimes W_j(E_6))\phi$$

with $i = 0,\ldots,9$ the labels of the vertices of the A_{10} diagram and $j = 0,1,2$ corresponding, respectively, to $\{0\dot{\otimes}0, 0\dot{\otimes}3, 0\dot{\otimes}4\}$ vertices of $\mathrm{Oc}(E_6)$:

$$\begin{aligned}
Z_{00} = Z_{92} = {} & \textstyle\sum_{r=1}^{5} |\phi_{r,1}+\phi_{r,7}|^2 + |\phi_{r,4}+\phi_{r,8}|^2 + |\phi_{r,5}+\phi_{r,11}|^2 \\
Z_{10} = Z_{82} = {} & [\textstyle\sum_{r=1}^{4} (\bar{\phi}_{r,1}+\bar{\phi}_{r,7})(\phi_{r+1,1}+\phi_{r+1,7}) + (\bar{\phi}_{r,4}+\bar{\phi}_{r,8})(\phi_{r+1,4}+\phi_{r+1,8}) \\
& +(\bar{\phi}_{r,5}+\bar{\phi}_{r,11}) + (\phi_{r+1,5}+\phi_{r+1,11}) + (\bar{\phi}_{5,5}+\bar{\phi}_{5,11})(\phi_{5,1}+\phi_{5,7}) + hc] \\
& +|\phi_{5,4}+\phi_{5,8}|^2 \\
Z_{20} = Z_{72} = {} & \textstyle\sum_{r=2}^{5} |\phi_{r,1}+\phi_{r,7}|^2 + |\phi_{r,4}+\phi_{r,8}|^2 + |\phi_{r,5}+\phi_{r,11}|^2 + \sum_{r=1}^{3} (\bar{\phi}_{r,1}+\bar{\phi}_{r,7}) \\
& \times (\phi_{r+2,1}+\phi_{r+2,7}) + (\bar{\phi}_{r,4}+\bar{\phi}_{r,8})(\phi_{r+2,4}+\phi_{r+2,8}) + (\bar{\phi}_{r,5}+\bar{\phi}_{r,11}) \\
& \times (\phi_{r+2,5}+\phi_{r+2,11}) + (\bar{\phi}_{5,1}+\bar{\phi}_{5,7})(\phi_{4,1}+\phi_{4,7}) + (\bar{\phi}_{5,4}+\bar{\phi}_{5,8}) \\
& \times (\phi_{4,4}+\phi_{4,8}) + (\bar{\phi}_{5,5}+\bar{\phi}_{5,11})(\phi_{4,5}+\phi_{4,11}) + hc \\
Z_{30} = Z_{62} = {} & \textstyle\sum_{r=0}^{2} (\bar{\phi}_{4,1}+\bar{\phi}_{4,7})(\phi_{2r+1,1}+\phi_{2r+1,7}) + (\bar{\phi}_{4,4}+\bar{\phi}_{4,8})(\phi_{2r+1,4}+\phi_{2r+1,8}) \\
& + (\bar{\phi}_{4,5}+\bar{\phi}_{4,11})(\phi_{2r+1,5}+\phi_{2r+1,11}) \\
& + \textstyle\sum_{r=1}^{2} (\bar{\phi}_{2,1}+\bar{\phi}_{2,7})(\phi_{2r+1,1}+\phi_{2r+1,7}) + (\bar{\phi}_{2,4}+\bar{\phi}_{2,8})(\phi_{2r+1,4}+\phi_{2r+1,8}) \\
& + (\bar{\phi}_{2,5}+\bar{\phi}_{2,11})(\phi_{2r+1,5}+\phi_{2r+1,11}) + (\bar{\phi}_{5,1}+\bar{\phi}_{5,7})(\phi_{3,5}+\phi_{3,11}) \\
& + (\bar{\phi}_{5,4}+\bar{\phi}_{5,8})(\phi_{3,4}+\phi_{3,8}) + (\bar{\phi}_{5,5}+\bar{\phi}_{5,11})(\phi_{3,1}+\phi_{3,7}) \\
& + \textstyle\sum_{r=4}^{5} (\bar{\phi}_{r,1}+\bar{\phi}_{r,7})(\phi_{r,5}+\phi_{r,11}) + hc + \sum_{r=4}^{5} |\phi_{r,4}+\phi_{r,8}|^2 \\
Z_{40} = Z_{52} = {} & \textstyle\sum_{r=3}^{5} |\phi_{r,1}+\phi_{r,7}|^2 + |\phi_{r,4}+\phi_{r,8}|^2 + |\phi_{r,5}+\phi_{r,11}|^2 \\
& + \textstyle\sum_{r=1,3} (\bar{\phi}_{5,1}+\bar{\phi}_{5,7})(\phi_{r,1}+\phi_{r,7}) + (\bar{\phi}_{5,4}+\bar{\phi}_{5,8})(\phi_{r,4}+\phi_{r,8}) \\
& + (\bar{\phi}_{5,5}+\bar{\phi}_{5,11})(\phi_{r,5}+\phi_{r,11}) + (\bar{\phi}_{4,1}+\bar{\phi}_{4,7})(\phi_{2,1}+\phi_{2,7}) \\
& + (\bar{\phi}_{4,4}+\bar{\phi}_{4,8})(\phi_{2,4}+\phi_{2,8}) + (\bar{\phi}_{4,5}+\bar{\phi}_{4,11})(\phi_{2,5}+\phi_{2,11}) \\
& + \textstyle\sum_{r=2}^{4} (\bar{\phi}_{5,1}+\bar{\phi}_{5,7})(\phi_{r,5}+\phi_{r,11}) + (\bar{\phi}_{5,4}+\bar{\phi}_{5,8})(\phi_{r,4}+\phi_{r,8}) \\
& + (\bar{\phi}_{5,5}+\bar{\phi}_{5,11})(\phi_{r,1}+\phi_{r,7}) + hc
\end{aligned}$$

$$
\begin{aligned}
Z_{50} = Z_{42} = {} & \textstyle\sum_{r=3}^{5} |\phi_{r,4} + \phi_{r,8}|^2 + ((\bar\phi_{r,1} + \bar\phi_{r,7})(\phi_{r,5} + \phi_{r,11}) + hc) \\
& + \textstyle\sum_{r=2,4} (\bar\phi_{5,1} + \bar\phi_{5,7})(\phi_{r,1} + \phi_{r,7}) + (\bar\phi_{5,4} + \bar\phi_{5,8})(\phi_{r,4} + \phi_{r,8}) \\
& + (\bar\phi_{5,5} + \bar\phi_{5,11})(\phi_{r,5} + \phi_{r,11}) + \textstyle\sum_{r=1,3} (\bar\phi_{5,1} + \bar\phi_{5,7})(\phi_{r,5} + \phi_{r,11}) \\
& + (\bar\phi_{5,4} + \bar\phi_{5,8})(\phi_{r,4} + \phi_{r,8}) + (\bar\phi_{5,5} + \bar\phi_{5,11})(\phi_{r,1} + \phi_{r,7}) \\
& + (\bar\phi_{4,5} + \bar\phi_{4,11})((\phi_{2,1} + \phi_{2,7}) + (\phi_{3,5} + \phi_{3,11})) \\
& + (\bar\phi_{4,1} + \bar\phi_{4,7})((\phi_{2,5} + \phi_{2,11}) + (\phi_{3,1} + \phi_{3,7})) \\
& \times \textstyle\sum_{r=1,3} (\bar\phi_{4,4} + \bar\phi_{4,8})(\phi_{r,4} + \phi_{r,8}) + hc \\
Z_{60} = Z_{32} = {} & \textstyle\sum_{r=4}^{5} |\phi_{r,1} + \phi_{r,7}|^2 + |\phi_{r,4} + \phi_{r,8}|^2 + |\phi_{r,5} + \phi_{r,11}|^2 + (\bar\phi_{5,1} + \bar\phi_{5,7}) \\
& \times (\phi_{3,1} + \phi_{3,7}) + (\bar\phi_{5,4} + \bar\phi_{5,8})(\phi_{3,4} + \phi_{3,8}) + (\bar\phi_{5,5} + \bar\phi_{5,11})(\phi_{3,5} + \phi_{3,11}) \\
& + \textstyle\sum_{r=1,3,5} (\bar\phi_{4,1} + \bar\phi_{4,7})(\phi_{r,5} + \phi_{r,11}) + (\bar\phi_{4,4} + \bar\phi_{4,8})(\phi_{r,4} + \phi_{r,8}) \\
& + (\bar\phi_{4,5} + \bar\phi_{4,11})(\phi_{r,1} + \phi_{r,7}) + hc \\
Z_{70} = Z_{22} = {} & \textstyle\sum_{r=2}^{5} |\phi_{r,4} + \phi_{r,8}|^2 + (\bar\phi_{r,1} + \bar\phi_{r,7})(\phi_{r,5} + \phi_{r,11}) \\
& + \textstyle\sum_{r=1}^{3} (\bar\phi_{r,1} + \bar\phi_{r,7})(\phi_{r+2,5} + \phi_{r+2,11}) + (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r+2,4} + \phi_{r+2,8}) \\
& + (\bar\phi_{r,5} + \bar\phi_{r,11})(\phi_{r+2,1} + \phi_{r+2,7}) + (\bar\phi_{5,1} + \bar\phi_{5,7})(\phi_{4,1} + \phi_{4,7}) \\
& + (\bar\phi_{5,4} + \bar\phi_{5,8})(\phi_{4,4} + \phi_{4,8}) + (\bar\phi_{5,5} + \bar\phi_{5,11})(\phi_{4,5} + \phi_{4,11}) + hc \\
Z_{80} = Z_{12} = {} & \textstyle\sum_{r=1}^{4} (\bar\phi_{r,1} + \bar\phi_{r,7})(\phi_{r+1,5} + \phi_{r+1,11}) + (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r+1,4} + \phi_{r+1,8}) \\
& + (\bar\phi_{r,5} + \bar\phi_{r,11})(\phi_{r+1,1} + \phi_{r+1,7}) + hc + |\phi_{5,1} + \phi_{5,7}|^2 + |\phi_{5,4} + \phi_{5,8}|^2 \\
& + |\phi_{5,5} + \phi_{5,11}|^2 \\
Z_{90} = Z_{02} = {} & \textstyle\sum_{r=1}^{5} |\phi_{r,4} + \phi_{r,8}|^2 + ((\bar\phi_{r,1} + \bar\phi_{r,7})(\phi_{r,5} + \phi_{r,11}) + hc)
\end{aligned}
$$

$$
\begin{aligned}
Z_{01} = Z_{91} = {} & \textstyle\sum_{r=1}^{5} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r,5} + \phi_{r,11} + \phi_{r,1} + \phi_{r,7}) + hc \\
Z_{11} = Z_{81} = {} & \textstyle\sum_{r=1}^{4} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r+1,5} + \phi_{r+1,11} + \phi_{r+1,1} + \phi_{r+1,7}) + (r \leftrightarrow r+1) \\
& + hc + (\bar\phi_{5,4} + \bar\phi_{5,8})(\phi_{5,5} + \phi_{5,11} + \phi_{5,1} + \phi_{5,7}) + hc \\
Z_{21} = Z_{71} = {} & \textstyle\sum_{r=2}^{5} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r,5} + \phi_{r,11} + \phi_{r,1} + \phi_{r,7}) + hc \\
& + \textstyle\sum_{r=1}^{3} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r+2,5} + \phi_{r+2,11} + \phi_{r+2,1} + \phi_{r+2,7}) \\
& + (r \leftrightarrow r+2) + hc + (\bar\phi_{3,4} + \bar\phi_{3,8})(\phi_{4,5} + \phi_{4,11} + \phi_{4,1} + \phi_{4,7}) \\
& + (3 \leftrightarrow 4) + hc \\
Z_{31} = Z_{61} = {} & \textstyle\sum_{r=4}^{5} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r,5} + \phi_{r,11} + \phi_{r,1} + \phi_{r,7}) + hc \\
& + \textstyle\sum_{r=2,4} \sum_{s=1,2} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{2s+1,5} + \phi_{2s+1,11} + \phi_{2s+1,1} + \phi_{2s+1,7}) \\
& + (r \leftrightarrow 2s+1) + hc + (\bar\phi_{3,4} + \bar\phi_{3,8})(\phi_{5,5} + \phi_{5,11} + \phi_{5,1} + \phi_{5,7}) \\
& + (3 \leftrightarrow 5) + hc + (\bar\phi_{1,4} + \bar\phi_{1,8})(\phi_{4,5} + \phi_{4,11} + \phi_{4,1} + \phi_{4,7}) + (1 \leftrightarrow 4) + hc \\
Z_{41} = Z_{51} = {} & \textstyle\sum_{r=3}^{5} (\bar\phi_{r,4} + \bar\phi_{r,8})(\phi_{r,5} + \phi_{r,11} + \phi_{r,1} + \phi_{r,7}) + hc \\
& + \textstyle\sum_{r=1}^{4} (\bar\phi_{5,4} + \bar\phi_{5,8})(\phi_{r,5} + \phi_{r,11} + \phi_{r,1} + \phi_{r,7}) + (5 \leftrightarrow r) + hc \\
& + \textstyle\sum_{r=2}^{3} (\bar\phi_{4,4} + \bar\phi_{4,8})(\phi_{r,5} + \phi_{r,11} + \phi_{r,1} + \phi_{r,7}) + (4 \leftrightarrow r) + hc
\end{aligned}
$$

5.2. *Examples from higher Coxeter–Dynkin system*

In general, a pair of generalized Dynkin diagrams (Di Francesco–Zuber diagrams in the case of the SU(3) system) and of levels k_1 and k_2 can be associated with a conformal theory whose central charge was recalled in the previous section. For the SU(3) case, the

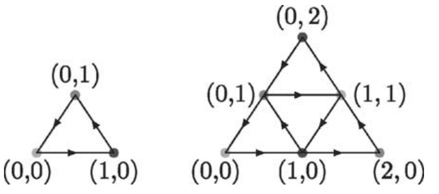

Fig. 7. The $\mathcal{A}_1$ and $\mathcal{A}_2$ generalized Dynkin.

corresponding generalized dual Coxeter numbers or altitudes are obtained from the relation $k=\kappa-3$. Taking $k_1+1=k_2$, like in the Virasoro minimal models, leads to a series of unitary $\mathcal{W}_3$-minimal models with central charges $4/5, 6/5, 10/7, 11/7, 5/3, 26/15, 98/55, 20/11, 24/13, \ldots$. In what follows we discuss two of these unitary theories corresponding to the $(\mathcal{A}_1, \mathcal{A}_2)$ and the $(\mathcal{A}_4, \mathcal{E}_5)$ pairs.

5.2.1. The $(\mathcal{A}_1, \mathcal{A}_2)$ model

Toric matrices and partition functions. The first member of this series, $c=4/5$, corresponds to the $(\mathcal{A}_1, \mathcal{A}_2)$ pair of diagrams with generalized Coxeter numbers ($\kappa_{\mathcal{A}_1} = 4, \kappa_{\mathcal{A}_2=5}$) (Fig. 7). We know that there are already two minimal (i.e., $\mathcal{W}_2$-minimal) models associated with this value of central charge, the diagonal (A_4, A_5) theory and the three states Potts model (A_4, D_4) already discussed in the previous section. We start considering the toric matrices of type $W_{\mathcal{A}_k}(\lambda, 00) = W_{\mathcal{A}_k}(\lambda)$ with $k=1,2$ and λ the weight of the representation (also index of the vertices of the diagram), $\lambda_{\mathcal{A}_1} = \{(00),(10),(01)\}$ and $\lambda_{\mathcal{A}_2} = \{(00),(11),(02),(10),(01),(20)\}$. Triality θ is obviously defined on these two diagrams in the following way: $\mathcal{A}_1 : \theta(00)=0, \theta(10)=1, \theta(01)=2$ and $\mathcal{A}_2 : \theta(00)=\theta(10)=0, \theta(01)=\theta(20)=2, \theta(10)=\theta(02)=1$. We have three toric matrices (also graph matrices in this case) for $\mathcal{A}_1$:

$$W_{\mathcal{A}_1}(00) \equiv W_{\mathcal{A}_1}[1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad W_{\mathcal{A}_1}(10) \equiv W_{\mathcal{A}_1}[2] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$W_{\mathcal{A}_1}(01) \equiv W_{\mathcal{A}_1}[3] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1}$$

and six toric matrices (also graph matrices in this case) for $\mathcal{A}_2$:

$$W_{\mathcal{A}_2}(00) \equiv W_{\mathcal{A}_2}[1] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$W_{\mathcal{A}_2}(10) \equiv W_{\mathcal{A}_2}[2] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} W_{\mathcal{A}_2}(01) &\equiv W_{\mathcal{A}_2}[3] = (W_{\mathcal{A}_2}(10))^{\mathrm{T}}, \\ W_{\mathcal{A}_2}(20) &\equiv W_{\mathcal{A}_2}[4] = W_{\mathcal{A}_2}(10) \cdot W_{\mathcal{A}_2}(10) - W_{\mathcal{A}_2}(01), \\ W_{\mathcal{A}_2}(02) &\equiv W_{\mathcal{A}_2}[5] = (W_{\mathcal{A}_2}(20))^{\mathrm{T}}, \\ W_{\mathcal{A}_2}(11) &\equiv W_{\mathcal{A}_2}[6] = W_{\mathcal{A}_2}(10) \cdot W_{\mathcal{A}_2}(01) - W_{\mathcal{A}_2}(00). \end{aligned}$$

The fundamental twisted partition functions are $Z(\lambda, \nu) = (1/3)\bar{\chi} W^{(\mathcal{A}_1, \mathcal{A}_2)}(\lambda, \nu)\chi$, with $W^{(\mathcal{A}_1 \mathcal{A}_2)}[\lambda, \nu] = W^{(\mathcal{A}_1)}[\lambda] \otimes W^{(\mathcal{A}_2)}[\nu]$, a matrix of dimension $(3 \times 6)^2$, and where $\chi = \chi[\lambda^{\mathcal{A}_1}, \mu^{\mathcal{A}_2}]$ denotes the basis

$$\begin{aligned} \{&\chi[00, 00], \chi[10, 00], \chi[01, 00], \chi[00, 11], \chi[10, 11], \chi[01, 11], \chi[00, 02], \\ &\chi[10, 02], \chi[01, 02], \chi[00, 10], \chi[10, 10], \chi[01, 10], \chi[00, 01], \chi[10, 01], \\ &\chi[01, 01], \chi[00, 20], \chi[10, 20], \chi[01, 20]\}. \end{aligned}$$

The matrix $W^{(\mathcal{A}_1 \mathcal{A}_2)}(00, 00) \equiv W^{(\mathcal{A}_1 \mathcal{A}_2)}[1, 1] W^{(\mathcal{A}_1)}[1] \otimes W^{(\mathcal{A}_2)}[1]$ is the modular invariant; in the base $\{\chi\}$, it is just the identity matrix of size $(18, 18)$. We give another example, $W^{(\mathcal{A}_1, \mathcal{A}_2)}(00, 10) \equiv W^{(\mathcal{A}_1 \mathcal{A}_2)}[1, 2] \doteq W^{(\mathcal{A}_1)}[1] \otimes W^{(\mathcal{A}_2)}[2]$ which is one of the twisted mass matrices:

$$W^{(\mathcal{A}_1, \mathcal{A}_2)}[1, 2] = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 \\
\hline
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & .
\end{array}\right)$$

As in the previous section, we are considering only the "fundamental toric matrices", i.e., those of type $W^{(\mathcal{A}_1,\mathcal{A}_2)}[\lambda,\nu] = W^{(\mathcal{A}_1)}[\lambda,00]\otimes W^{(\mathcal{A}_2)}[\nu,00]$. The more general ones would be of the type $W^{(\mathcal{A}_1,\mathcal{A}_2)}[\lambda,\nu,\tau,\mu] = W^{(\mathcal{A}_1)}[\lambda,\nu]\otimes W^{(\mathcal{A}_2)}[\tau,\mu]$. A priori, we obtain in this way $3\times 6 = 18$ toric matrices, but because of the identifications resulting from the $\mathbb{Z}_3$ symmetries of the $\mathcal{W}_3$ Kac table, we obtain, at the end, only $18/3 = 6$ distinct[26] partition functions $Z[i,j]$. The group $\mathbb{Z}_3$ acts on the pairs of vertices belonging to these two diagrams in a geometrically very intuitive way (counter-clockwise on both $\mathcal{A}_1$ and $\mathcal{A}_2$) so that characters are identified as follows:

$$\begin{aligned}
&\chi[00,00] = \chi[10,20] = \chi[01,02] = \chi_1,\\
&\chi[00,02] = \chi[10,00] = \chi[01,20] = \chi_2,\\
&\chi[00,20] = \chi[10,02] = \chi[01,00] = \chi_3,\\
&\chi[00,11] = \chi[10,01] = \chi[01,10] = \chi_4,\\
&\chi[00,10] = \chi[10,11] = \chi[01,01] = \chi_5,\\
&\chi[00,01] = \chi[10,10] = \chi[01,11] = \chi_6.
\end{aligned}$$

Implementation of this Z_3 symmetry over the characters leads to the following table where we list the six partition functions of the form $Z[i,j]$:

$$\begin{aligned}
&Z[1,1] = Z[2,4] = 3Z[3,5] = \textstyle\sum_{i=1}^{6}|\chi_i|^2\\
&Z[1,2] = Z[2,6] = 3Z[3,3] = \chi_1\bar\chi_5 + \chi_2\bar\chi_6 + \chi_3\bar\chi_4 + \chi_4(\bar\chi_2+\bar\chi_5) + \chi_5(\bar\chi_3+\bar\chi_6)\\
&\qquad\qquad + \chi_6(\bar\chi_1+\bar\chi_4)\\
&Z[1,3] = Z[2,2] = 3Z[3,6] = \chi_1\bar\chi_6 + \chi_2\bar\chi_4 + \chi_3\bar\chi_5 + \chi_4(\bar\chi_3+\bar\chi_6)\\
&\qquad\qquad + \chi_5(\bar\chi_1+\bar\chi_4) + \chi_6(\bar\chi_2+\bar\chi_5)\\
&Z[1,4] = Z[2,5] = 3Z[3,1] = \chi_1\bar\chi_3 + \chi_2\bar\chi_1 + \chi_3\bar\chi_2 + \chi_4\bar\chi_6 + \chi_5\bar\chi_4 + \chi_6\bar\chi_5\\
&Z[1,5] = Z[2,1] = 3Z[3,4] = \bar Z[1,4]\\
&Z[1,6] = Z[2,3] = 3Z[3,2] = |\chi_1+\chi_4|^2 + |\chi_2+\chi_5|^2 + |\chi_3+\chi_6|^2 + \textstyle\sum_{i=4}^{6}|\chi_i|^2
\end{aligned}$$

The Potts model recovered. From the value of the central charge (4/5) it is expected that the present model is the Potts model in a new guise. It is indeed so and this has been known for quite a while. However, here we want to show that not only we recover the usual (undeformed) partition function, but also the whole set of (four) twisted partition functions that were determined in Section 4.3 and denoted in boldface. We first calculate conformal weights for the SU(3) fields χ from the generalized $\mathcal{W}_3$ Rocha–Cariddi recalled in Section 4.3. Remember that r, s labels are shifted by $(1,1)$ compared with (λ,μ) labels. One finds: $h(\chi[00,00]) = 0$ and this is compatible[27] with the SU(2) fields for which $h = 0$ or 3, i.e., ϕ_{11} and ϕ_{41}; $h(\chi[00,11]) = 2/5$, compatible with the SU(2) fields for which $h = 2/5$ or $h = 7/5 = (2/5)+1$, i.e., ϕ_{21} and ϕ_{31}; $h(\chi[00,02]) = 2/3$, compatible with ϕ_{13}, and $h(\chi[00,20]) = 2/3$, also compatible with ϕ_{13}; $h(\chi[00,10]) = 1/15$, compatible

[26] For a $\mathcal{W}_3$-minimal model of type $(\mathcal{A}_k,\mathcal{A}_{k+1})$, we would obtain $(k+1)(k+2)^2(k+3)/12$ distinct functions.

[27] Compatibility of weights of SU(2) versus SU(3) is only meaningful modulo integers.

with ϕ_{23} and $h(\chi[00, 01]) = 1/15$, also compatible with ϕ_{23}. It is therefore natural to consider the branching rules:

$$\chi[00,00] \to \phi_{11} + \phi_{41}, \qquad \chi[00,11] \to \phi_{21} + \phi_{31}, \qquad \chi[00,02] \to \phi_{13},$$
$$\chi[00,20] \to \phi_{13}, \qquad \chi[00,10] \to \phi_{23}, \qquad \chi[00,01] \to \phi_{23}. \tag{2}$$

If we now perform these substitutions in the twisted partition functions of the previous table, we find

$$Z[11] \to Z_0, \qquad Z[12] \to 2Z_3, \qquad Z[13] \to 2Z_2, \qquad Z[14] \to Z1,$$
$$Z[15] \to \overline{Z1}, \qquad Z[16] \to Z0 + Z2.$$

The six twisted partitions functions of this SU(3) minimal model can therefore be reinterpreted in terms of the four twisted partitions functions of the SU(2) Potts model obtained in Section 5.1.2.

5.2.2. The $(\mathcal{A}_4, \mathcal{E}_5)$ *model*

The $(\mathcal{A}_4, \mathcal{E}_5)$ pair of diagrams with generalized Coxeter numbers ($\kappa_{\mathcal{A}_4} = 7, \kappa_{\mathcal{E}_5} = 8$) corresponds to a $\mathcal{W}_3$-minimal and unitary conformal model, with central charge $c = 11/7$. One of these diagrams ($\mathcal{E}_5$) was displayed in Section 3.2, and $\mathcal{A}_4$ is of course similar to $\mathcal{A}_5$ (displayed in the same section) but with only four levels. There are several possible theories associated to this value of central charge, one is the diagonal theory associated to the pair $(\mathcal{A}_4, \mathcal{A}_5)$, another is the one we are considering here. There are 15 toric matrices (also graph matrices in this case) of type $W_{\mathcal{A}_4}(\lambda, 00) = W_{\mathcal{A}_4}(\lambda)$ with λ labeling the vertices of the generalized Dynkin diagram, $\lambda_{\mathcal{A}_4} = \{(00), (30), (03), (11), (22), (10), (40), (21), (02), (13), (20), (12), (04), (01), (31)\}$ and 24 of type $W_{\mathcal{E}_5}(\sigma, 00) = W_{\mathcal{E}_5}(\sigma)$ with $\sigma = \rho \times \nu$ labels of the $\mathrm{Oc}(\mathcal{E}_5)$ vertices (this graph was obtained in [12] and is recalled below 8). Here $\rho, \nu \in \lambda_{\mathcal{E}_5} = \{1_0, 1_3, 2_3, 2_0, 1_2, 1_5, 2_2, 2_5, 1_1, 1_4, 2_1, 2_4\}$ labels the vertices of the generalized $\mathcal{E}_5$ Dynkin diagram. The $W_{\mathcal{E}_5}(\sigma)$ are matrices of dimension 21×21 whose entries (i, j) corresponds to the vertices of the diagram $\mathcal{A}_5 = \mathcal{A}(\mathcal{E}_5)$.

The general twisted partition functions are given by

$$Z(\lambda, \sigma) = \tfrac{1}{3} \bar{\chi} W^{(\mathcal{A}_4, E_5)}(\lambda, \sigma) \chi,$$

where $\chi = \chi[\lambda^{\mathcal{A}_4}, \lambda^{\mathcal{A}_5}]$ and $W^{(\mathcal{A}_4\mathcal{E}_5)}[\lambda, \sigma] = W^{(\mathcal{A}_4)}[\lambda] \otimes W^{(\mathcal{A}_5)}[\sigma]$.

Exponents of $\mathcal{E}_5$ can be read, for example, from the modular invariant toric matrix given in Section 3.2. These are particular $\mathcal{A}_5$ vertices $s = (s_1, s_2)$ given by the list $\{(0,0), (2,2), (0,2), (3,2), (2,0), (2,3), (2,1), (0,5), (3,0), (0,3), (1,2), (5,0)\}$. The $\mathbb{Z}_3$ action on $\mathcal{A}_4$ gives, a priori, five equivalence classes labeled, for example by $\{(0,0), (0,1), (0,2), (0,3), (1,1)\}$. All together, the untwisted $(\mathcal{A}_4, \mathcal{E}_5)$ model will therefore involve in principle $12 \times 5 = 60$ distinct $\mathcal{W}_3$ characters with conformal weights given by the following table where we are including the 15 r vertices of $\mathcal{A}_4$ in a particular order $\{(00), (40), (04)\}, \{(22), (02), (20)\}, \{(03), (10), (31)\}, \{(13), (01), (30)\}, \{(11), (12), (21)\}$ to make manifest the occurrence of the five mentioned equivalence classes (this is only a subset of the Kac table of the pair $(\mathcal{A}_4, \mathcal{A}_5)$):

s	r														
	00	40	04	22	02	20	03	10	31	13	01	30	11	12	21
00	0	$\frac{20}{3}$	$\frac{20}{3}$	$\frac{36}{7}$	$\frac{38}{21}$	$\frac{38}{21}$	$\frac{27}{7}$	$\frac{11}{21}$	$\frac{116}{21}$	$\frac{116}{21}$	$\frac{11}{21}$	$\frac{27}{7}$	$\frac{10}{7}$	$\frac{65}{21}$	$\frac{65}{21}$
05	$\frac{20}{3}$	$\frac{20}{3}$	0	$\frac{38}{21}$	$\frac{38}{21}$	$\frac{36}{7}$	$\frac{11}{21}$	$\frac{116}{21}$	$\frac{27}{7}$	$\frac{11}{21}$	$\frac{27}{7}$	$\frac{116}{21}$	$\frac{65}{21}$	$\frac{10}{7}$	$\frac{65}{21}$
50	$\frac{20}{3}$	0	$\frac{20}{3}$	$\frac{38}{21}$	$\frac{36}{7}$	$\frac{38}{21}$	$\frac{116}{21}$	$\frac{27}{7}$	$\frac{11}{21}$	$\frac{27}{7}$	$\frac{116}{21}$	$\frac{11}{21}$	$\frac{65}{21}$	$\frac{65}{21}$	$\frac{10}{7}$
03	$\frac{9}{4}$	$\frac{59}{12}$	$\frac{11}{12}$	$\frac{39}{28}$	$\frac{5}{84}$	$\frac{173}{84}$	$\frac{3}{28}$	$\frac{149}{84}$	$\frac{233}{84}$	$\frac{65}{84}$	$\frac{65}{84}$	$\frac{87}{28}$	$\frac{19}{28}$	$\frac{29}{84}$	$\frac{113}{84}$
20	$\frac{11}{12}$	$\frac{9}{4}$	$\frac{59}{12}$	$\frac{173}{84}$	$\frac{39}{28}$	$\frac{5}{84}$	$\frac{233}{84}$	$\frac{3}{28}$	$\frac{149}{84}$	$\frac{87}{28}$	$\frac{65}{84}$	$\frac{65}{84}$	$\frac{29}{84}$	$\frac{113}{84}$	$\frac{19}{28}$
32	$\frac{59}{12}$	$\frac{11}{12}$	$\frac{9}{4}$	$\frac{5}{84}$	$\frac{173}{84}$	$\frac{39}{28}$	$\frac{149}{84}$	$\frac{233}{84}$	$\frac{3}{28}$	$\frac{65}{84}$	$\frac{87}{28}$	$\frac{65}{84}$	$\frac{113}{84}$	$\frac{19}{28}$	$\frac{29}{84}$
30	$\frac{9}{4}$	$\frac{11}{12}$	$\frac{59}{12}$	$\frac{39}{28}$	$\frac{173}{84}$	$\frac{5}{84}$	$\frac{87}{28}$	$\frac{65}{84}$	$\frac{65}{84}$	$\frac{233}{84}$	$\frac{149}{84}$	$\frac{3}{28}$	$\frac{19}{28}$	$\frac{113}{84}$	$\frac{29}{84}$
23	$\frac{59}{12}$	$\frac{9}{4}$	$\frac{11}{12}$	$\frac{5}{84}$	$\frac{39}{28}$	$\frac{173}{84}$	$\frac{65}{84}$	$\frac{87}{28}$	$\frac{65}{84}$	$\frac{3}{28}$	$\frac{233}{84}$	$\frac{149}{84}$	$\frac{113}{84}$	$\frac{29}{84}$	$\frac{19}{28}$
02	$\frac{11}{12}$	$\frac{59}{12}$	$\frac{9}{4}$	$\frac{173}{84}$	$\frac{5}{84}$	$\frac{39}{28}$	$\frac{65}{84}$	$\frac{65}{84}$	$\frac{87}{28}$	$\frac{149}{84}$	$\frac{3}{28}$	$\frac{233}{84}$	$\frac{29}{84}$	$\frac{19}{28}$	$\frac{113}{84}$
22	3	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{1}{7}$	$\frac{17}{21}$	$\frac{17}{21}$	$\frac{6}{7}$	$\frac{32}{21}$	$\frac{11}{21}$	$\frac{11}{21}$	$\frac{32}{21}$	$\frac{6}{7}$	$\frac{3}{7}$	$\frac{2}{21}$	$\frac{2}{21}$
12	$\frac{5}{3}$	3	$\frac{5}{3}$	$\frac{17}{21}$	$\frac{1}{7}$	$\frac{17}{21}$	$\frac{11}{21}$	$\frac{6}{7}$	$\frac{32}{21}$	$\frac{6}{7}$	$\frac{11}{21}$	$\frac{32}{21}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{3}{7}$
21	$\frac{5}{3}$	$\frac{5}{3}$	3	$\frac{17}{21}$	$\frac{17}{21}$	$\frac{1}{7}$	$\frac{32}{21}$	$\frac{11}{21}$	$\frac{6}{7}$	$\frac{32}{21}$	$\frac{6}{7}$	$\frac{11}{21}$	$\frac{2}{21}$	$\frac{3}{7}$	$\frac{2}{21}$

The Ocneanu graph of $\mathcal{E}_5$ has 24 points and the intersection of the vector spaces spanned by the 12 left and the 12 right generators—ambichiral subspace—is of dimension 6 (generators are of the type $1_0\dot{\otimes}1_j = 1_j\dot{\otimes}1_0$) (Fig. 8). The supplementary subspace has also dimension 6. We therefore expect to obtain four sets of 5×6 twisted fundamental partition functions.

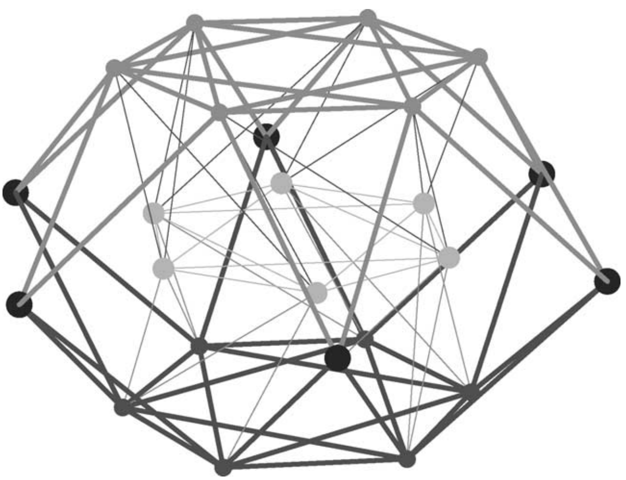

Fig. 8. Ocneanu graph for $\mathcal{E}_5$.

If we further restrict our attention to those fundamental twisted partition functions which only involve the fields appearing in the undeformed $(\mathcal{A}_4, \mathcal{E}_5)$ model (labeled by the above 60 conformal weights), i.e., if we only take the ambichiral points into account, we expect $5 \times 6 = 30$ distinct cases. Notice that the description of this $\mathcal{W}_3$-minimal model is very similar to the one that we made for the Virasoro minimal model of type (A_{10}, E_6).

We shall not give this full list of 30 partition functions but only two of them: those associated with toric matrices $W_{00,00}(\mathcal{A}_4) \otimes W_{1_0\otimes 1_0}(\mathcal{E}_5)$ and $W_{00,00}(\mathcal{A}_4) \otimes W_{1_0\otimes 1_3}(\mathcal{E}_5)$:

$$
\begin{aligned}
Z[00, 1_0 \times 1_0] = \textstyle\sum_i & |\chi[i, 05] + \chi[i, 21]|^2 + |\chi[i, 00] + \chi[i, 22]|^2 + |\chi[i, 20] + \chi[i, 23]|^2 \\
& + |\chi[i, 03] + \chi[i, 30]|^2 + |\chi[i, 02] + \chi[i, 32]|^2 + |\chi[i, 12] + \chi[i, 50]|^2 \\
Z[00, 1_0 \times 1_3] = \textstyle\sum_i & (\chi[i, 02] + \chi[i, 32])(\bar{\chi}[i, 05] + \bar{\chi}[i, 21]) + (\chi[i, 03] + \chi[i, 30])(\bar{\chi}[i, 00] \\
& + \bar{\chi}[i, 22]) + (\chi[i, 12] + \chi[i, 50])(\bar{\chi}[i, 20] + \bar{\chi}[i, 23]) + hc
\end{aligned}
$$

where the sums are over $i = (0, 0), (0, 1), (0, 2), (0, 3), (1, 1)$ vertices of $\mathcal{A}_4$.

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